

# Gradient estimates for heat kernels and harmonic functions

Renjin Jiang, Thierry Coulhon, Pekka Koskela and Adam Sikora

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**Abstract.** Let  $(X, d, \mu)$  be a doubling metric measure space endowed with a Dirichlet form  $\mathcal{E}$  deriving from a “carré du champ”. Assume that  $(X, d, \mu, \mathcal{E})$  supports a scale-invariant  $L^2$ -Poincaré inequality. In this article, we study the following properties of harmonic functions, heat kernels and Riesz transform for  $p \in (2, \infty]$ :

- (i)  $(G_p)$ :  $L^p$ -boundedness of the gradient of the associated heat semigroup;
- (ii)  $(RH_p)$ :  $L^p$ -reverse Hölder inequality for the gradient of harmonic functions;
- (iii)  $(R_p)$ :  $L^p$ -boundedness of the Riesz transform ( $p < \infty$ );
- (iv)  $(GBE)$ : a generalized Bakry-Émery condition.

We show that, for  $p \in (2, \infty)$ , (i), (ii) (iii) are equivalent, while for  $p = \infty$ , (i), (ii), (iv) are equivalent. Moreover, some of these equivalences still hold under weaker conditions than the  $L^2$ -Poincaré inequality.

Our result gives for  $p = \infty$  a characterisation of Li-Yau’s gradient estimate of heat kernels and Yau type gradient estimate of harmonic functions, while for  $p \in (2, \infty)$  it is a substantial improvement as well as a generalisation of earlier results by Auscher-Coulhon-Duong-Hofmann [7] and Auscher-Coulhon [6]. Applications to isoperimetric inequalities and Sobolev inequalities are given. Our results apply to Riemannian and sub-Riemannian manifolds as well as non-smooth spaces, and degenerate elliptic/parabolic equations in these settings. Our results also admit local versions.

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# 1 Introduction

## 1.1 Background and main results

On complete Riemannian manifolds and on more general metric measure spaces endowed with a Dirichlet form, Gaussian heat kernels upper and lower estimates are well understood since the works of Saloff-Coste [93], Grigor'yan [52], Sturm [99, 100, 101], see also [15, 58] and references therein. Together these estimates imply the doubling volume property and the Hölder regularity of the heat kernel (see [42] for a new and direct proof of the latter fact). A fundamental and non-trivial consequence of the known characterisation of these estimates, in terms of the volume

doubling property and a scale-invariant  $L^2$ -Poincaré inequality, is that they are stable under quasi-isometries.

By contrast, the matching upper estimate of the gradient of the heat kernel is only known to hold in very specific cases : on manifolds with non-negative Ricci curvature [82], on Lie groups with polynomial volume growth [91], and on covering manifolds with polynomial volume growth [38, 39]. There were also many efforts to derive upper bounds of the gradient of the heat kernel, but only for small time (local result) unless one assumes non-negativity of the curvature, by using probabilistic methods including coupling and derivation of Bismut type formulae; see [36, 87, 89, 98, 104] and references therein.

No handy global characterisation exists (see however [35, Theorem 4.2]). Note that no equivalent property can exhibit invariance under quasi-isometry : the example of divergence form operators with bounded measurable coefficients shows that the Lipschitz character of the heat kernel is not generic and not stable under quasi-isometry. However, non-negative curvature is too restrictive a condition, since it is very unstable under any kind of perturbation. Moreover, it is desirable to find a common reason that would explain why the property holds in the above three families of examples. Such a condition was introduced in [69, Theorem 3.2] and [71, Theorem 3.1], where it is proven that a certain quantitative Lipschitz regularity of Cheeger-harmonic functions implies an upper estimate of the gradient of the heat kernel, and it is not hard to obtain such regularity of harmonic functions in the aforementioned settings, see Section 8.

In the present paper, first we give a converse to this implication, second we give an  $L^p$  version of this equivalence which can be seen as an  $L^\infty$  one. Indeed, an important motivation to study pointwise estimates of the gradient of the heat kernel is that they open the way to the boundedness of Riesz transforms on  $L^p$  for all  $p \in (1, +\infty)$  (see [7]). Further, it appeared in [7] that a weaker,  $L^p$  version of these estimates governs the boundedness of Riesz transforms on  $L^p$  in a interval  $(2, p_0)$ , for  $2 < p_0 < +\infty$ . Details will be given below.

To summarise, we give characterisations of these pointwise and integrated gradient of the heat kernel estimates in terms of harmonic functions. In other words, we eliminate time. This is a first step towards a geometric understanding of these estimates.

Let  $X$  be a locally compact, separable, metrisable, and connected space equipped with a Borel measure  $\mu$  that is finite on compact sets and strictly positive on non-empty open sets. Consider a strongly local and regular Dirichlet form  $\mathcal{E}$  on  $L^2(X, \mu)$  with dense domain  $\mathcal{D} \subset L^2(X, \mu)$  (see [49] or [55] for precise definitions). According to Beurling-Deny [17], such a form can be written as

$$\mathcal{E}(f, g) = \int_X d\Gamma(f, g)$$

for all  $f, g \in \mathcal{D}$ , where  $\Gamma$  is a measure-valued non-negative and symmetric bilinear form defined by the formula

$$\int_X \varphi d\Gamma(f, g) := \frac{1}{2} [\mathcal{E}(f, \varphi g) + \mathcal{E}(g, \varphi f) - \mathcal{E}(fg, \varphi)]$$

for all  $f, g \in \mathcal{D} \cap L^\infty(X, \mu)$  and  $\varphi \in \mathcal{D} \cap \mathcal{C}_0(X)$ . Here and in what follows,  $\mathcal{C}(X)$  denotes the collection of all continuous functions on  $X$  and  $\mathcal{C}_0(X)$  denotes the set of all functions in  $\mathcal{C}(X)$  with compact support. We shall assume in addition that  $\mathcal{E}$  admits a “*carré du champ*”, meaning that

$\Gamma(f, g)$  is absolutely continuous with respect to  $\mu$ , for all  $f, g \in \mathcal{D}$ . In what follows, for simplicity of notation, we will denote by  $\langle \nabla f, \nabla g \rangle$  the energy density  $\frac{d\Gamma(f, g)}{d\mu}$ , and by  $|\nabla f|$  the square root of  $\frac{d\Gamma(f, f)}{d\mu}$ .

Since  $\mathcal{E}$  is strongly local,  $\Gamma$  is local and satisfies the Leibniz rule and the chain rule; see [49]. Therefore we can define  $\mathcal{E}(f, g)$  and  $\Gamma(f, g)$  locally. Denote by  $\mathcal{D}_{\text{loc}}$  the collection of all  $f \in L^2_{\text{loc}}(X)$  for which, for each relatively compact set  $K \subset X$ , there exists a function  $h \in \mathcal{D}$  such that  $f = h$  almost everywhere on  $K$ . The intrinsic (pseudo-)distance on  $X$  associated to  $\mathcal{E}$  is then defined by

$$d(x, y) := \sup \{f(x) - f(y) : f \in \mathcal{D}_{\text{loc}} \cap \mathcal{C}(X), |\nabla f| \leq 1 \text{ a.e.}\}.$$

In this paper, we always assume that  $d$  is indeed a distance (meaning that for  $x \neq y$ ,  $0 < d(x, y) < +\infty$ ) and that the topology induced by  $d$  is equivalent to the original topology on  $X$ . Moreover, we assume that  $(X, d)$  is a complete metric space. Under this assumption,  $(X, d)$  is a geodesic length space; see for instance [4, 55, 99].

To summarize the above situation, we shall say that  $(X, d, \mu, \mathcal{E})$  is a metric measure Dirichlet space endowed with a “*carré du champ*”. Define  $W^{1,2}(X)$  as the domain of  $\mathcal{E}$  with the norm

$$\|f\|_{W^{1,2}(X)} = \|f\|_2 + \mathcal{E}(f, f).$$

For  $p \geq 2$  and an open set  $U \subset X$  ( $U$  may equal  $X$ ), the local Sobolev space  $W^{1,p}_{\text{loc}}(U)$  on  $U$  is defined to be the collection of all functions  $f$  satisfying  $f\varphi \in \mathcal{D} = W^{1,2}(X)$ , for each Lipschitz function  $\varphi$  with compact support in  $U$ , with  $f\varphi, |\nabla(f\varphi)| \in L^p(X, \mu)$ . The Sobolev space  $W^{1,p}(U)$ ,  $p > 2$ , is then defined as the collection of all functions  $f \in W^{1,p}_{\text{loc}}(U)$  satisfying  $f, |\nabla f| \in L^p(U)$ . The space  $W^{1,p}_0(U)$  is defined to be the closure in  $W^{1,p}(X)$  of functions in  $W^{1,p}(U)$  with compact support in  $U$ .

Corresponding to such a Dirichlet form  $\mathcal{E}$ , there exists an infinitesimal generator whose opposite we shall denote by  $\mathcal{L}$ , acting on a dense domain  $\mathcal{D}(\mathcal{L})$  in  $L^2(X, \mu)$ ,  $\mathcal{D}(\mathcal{L}) \subset W^{1,2}(X)$ , so that for all  $f \in \mathcal{D}(\mathcal{L})$  and each  $g \in W^{1,2}(X)$ ,

$$\int_X f(x) \mathcal{L}g(x) d\mu(x) = \mathcal{E}(f, g).$$

Let  $B(x, r)$  denotes the open ball with center  $x$  and radius  $r$  with respect to the distance  $d$ , and  $CB(x, r) := B(x, Cr)$ . For simplicity we shall denote  $V(x, r) := \mu(B(x, r))$  for  $x \in X$  and  $r > 0$ . We say that the metric measure space  $(X, d, \mu)$  satisfies the volume doubling property if there exists a constant  $C_D > 1$  such that for every  $x \in X$  and all  $0 < r$ ,

$$(D) \quad V(x, 2r) \leq C_D V(x, r).$$

It follows easily that there exist  $Q > 0$  and  $C_Q > 0$  depending only on  $C_D$  such that for every  $x \in X$  and all  $0 < r < R$ ,

$$(D_Q) \quad V(x, R) \leq C_Q \left(\frac{R}{r}\right)^Q V(x, r).$$

Notice that  $(X, d, \mu)$  satisfies  $(D)$  if and only if it satisfies  $(D_Q)$  for some  $Q > 0$ . Moreover, since  $(D_Q)$  implies  $(D_{\tilde{Q}})$  for each  $\tilde{Q} > Q$ , we shall assume without loss of generality that  $Q > 1$ .

A local Sobolev inequality  $(LS_q)$ ,  $q > 2$ , holds if for every ball  $B = B(x, r)$  and each  $f \in W_0^{1,2}(B)$ ,

$$(LS_q) \quad \left( \int_B |f|^q d\mu \right)^{2/q} \leq C_{LS} \left( \int_B |f|^2 d\mu + \frac{r^2}{V(x, r)} \mathcal{E}(f, f) \right).$$

Under condition  $(D)$ , it is known that  $(LS_q)$ , for some  $q > 2$ , is equivalent to the heat semigroup  $H_t = e^{-t\mathcal{L}}$  has a kernel  $h_t$ , called the heat kernel, which satisfies an upper Gaussian bound  $(UE)$  as

$$(UE) \quad h_t(x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp \left\{ -c \frac{d^2(x, y)}{t} \right\};$$

see [15, 21] for instance.

We say that  $(X, d, \mu, \mathcal{E})$  supports a local  $L^p$ -Poincaré inequality,  $p \in [2, \infty)$ , if for all  $r_0 > 0$  there exists  $C_P(r_0) > 0$  such that, for all  $0 < r < r_0$  and for every ball  $B = B(x_0, r)$  and each  $f \in W^{1,p}(B)$ ,

$$(P_{p, \text{loc}}) \quad \int_{B(x_0, r)} |f - f_B| d\mu \leq C_P(r_0) r \left( \int_{B(x_0, r)} |\nabla f|^p d\mu \right)^{1/p}.$$

Similarly,  $(P_{\infty, \text{loc}})$  requires

$$(P_{\infty, \text{loc}}) \quad \int_{B(x_0, r)} |f - f_B| d\mu \leq C_P(r_0) r \|\nabla f\|_{L^\infty(B)}.$$

Further, if there exists a constant  $C_P > 0$  such that the above inequalities hold for every ball  $B(x_0, r)$  and each  $f \in W^{1,p}(B)$  with  $C_P(r_0)$  replaced by  $C_P$ , then we say that  $(X, d, \mu)$  supports a scale-invariant  $L^p$ -Poincaré inequality,  $(P_p)$ ,  $p \in [2, \infty]$ .

According to [74] and that  $(X, d)$  is geodesic,  $(P_{p, \text{loc}})$  improves to an  $(P_{p-\varepsilon, \text{loc}})$  for some  $\varepsilon > 0$  whenever  $1 < p < \infty$ ; see also [72, p.403]. However in general  $(P_{\infty, \text{loc}})$  can not self-improve to  $(P_{q, \text{loc}})$  for any  $q < \infty$ ; see [40, 41]. We also note that,  $(P_2)$  together with  $(D)$  implies  $(LS_q)$  for some  $q \in (2, \infty]$ , but the converse is not true; see [21, 57].

By Sturm [99, 100, 101] (see Saloff-Coste [92, 93] and Grigor'yan [52] for earlier results on Riemannian manifolds), on a metric measure space  $(X, d, \mu)$  associated with a Dirichlet form  $\mathcal{E}$ ,  $(D)$  together with  $(P_2)$  are equivalent to the fact that the heat semigroup  $H_t = e^{-t\mathcal{L}}$  has a heat kernel  $h_t$ , that satisfies the Li-Yau estimate

$$(LY) \quad \frac{C^{-1}}{V(x, \sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{ct} \right\} \leq h_t(x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp \left\{ -c \frac{d^2(x, y)}{t} \right\}.$$

This was originally obtained in [82] on Riemannian manifolds with non-negative Ricci curvature. Moreover, the Li-Yau estimate is equivalent to a parabolic Harnack inequality for solutions to the

heat equation. The parabolic Harnack inequality obviously implies an elliptic Harnack inequality, which had been obtained earlier under doubling and Poincaré by Biroli-Mosco [18, 19]. Moreover, Hebisch and Saloff-Coste [58] (see also [15]) showed that an elliptic Harnack inequality also implies a parabolic one if one has  $(UE)$ . Notice that under  $(D)$ , the heat kernel is conservative, i.e.  $\int_X h_t(x, y) d\mu(y) = 1$  (see [100]).

However, in general,  $(D)$  and  $(UE)$ , or even  $(D)$  and  $(P_2)$  are sufficient neither for Lipschitz regularity of harmonic functions or heat kernels, nor for  $L^p$  ( $p > 2$ ) quantitative regularity of gradients of harmonic functions and heat kernels. This phenomenon already occurs for uniformly elliptic operators of divergence form in the Euclidean space, see for instance [22, 95] as well as Theorem 1.2, Theorem 1.6, Theorem 7.1 and Theorem 7.3 below.

Yau's gradient estimate for positive harmonic functions (cf. Yau [107], Cheng-Yau [31]) states that on non-compact Riemannian manifolds with Ricci curvature bounded below by  $-K$ ,  $K \geq 0$ , it holds

$$(Y_\infty) \quad \sup_{x \in B(x_0, r)} |\nabla \log u(x)| \leq C \left( \frac{1}{r} + \sqrt{K} \right),$$

for every ball  $B(x_0, r)$  and every positive harmonic function  $u$  in  $B(x_0, 2r)$ . Li-Yau's gradient estimate for heat kernels (c.f. Li-Yau [82]) on Riemannian manifolds with non-negative Ricci curvature states that

$$(GLY_\infty) \quad |\nabla_x h_t(x, y)| \leq \frac{C}{\sqrt{t}V(y, \sqrt{t})} \exp \left\{ -c \frac{d^2(x, y)}{t} \right\}.$$

These two gradient estimates are fundamental tools in geometric analysis and related fields, and there has been many efforts afterwards to generalize them to different settings, see [37, 38, 39, 43, 50, 63, 69, 79, 88, 89, 91, 108, 109, 110] for instance.

Let us review some of these generalisations. Saloff-Coste [91] obtained  $(GLY_\infty)$  on Lie groups with polynomial growth. Duney [38, 39] obtained  $(GLY_\infty)$  on Riemannian covering manifolds. On Heisenberg type groups, Driver and Melcher [37] and Hu-Li [63] obtained a Bakry-Émery type inequality, which implies  $(GLY_\infty)$ . Zhang [108] obtained Yau's gradient estimate ( $K = 0$ ) on Riemannian manifolds of non-negative Ricci curvature with a small perturbation. In recent years, in a sequence of works [14, 50, 68, 69, 109, 110], Yau's gradient estimate for harmonic functions and Li-Yau's gradient estimate for heat kernels (and their local versions) have been further generalized to metric measure spaces and graphs satisfying suitable curvature assumptions; we refer the reader to [3, 4, 26, 27, 28, 44, 62, 83, 102, 103] for recent developments of lower Ricci curvature bounds and related calculus on metric measure spaces. As we already said, our main aim in the present paper is to characterise heat kernel gradient bounds without making any curvature assumption.

The conjunction of [7, Theorem 1.4] and [35, Corollary 2.2] shows that  $(D)$  and  $(GLY_\infty)$  yield the boundedness of Riesz transforms on  $L^p$ :

$$(R_p) \quad \|\nabla \mathcal{L}^{-1/2} f\|_p \leq C \|f\|_p, \quad \forall f \in L^p(X, \mu)$$

for all  $p \in (1, +\infty)$ . On the other hand, under  $(D)$  and  $(UE)$ ,  $(GLY_\infty)$  is known to be equivalent to the boundedness of the gradient of the heat semigroup

$$(G_\infty) \quad |||\nabla H_t|||_{\infty \rightarrow \infty} \leq \frac{C}{\sqrt{t}}$$

(see [7, p.919] and [34, Theorem 4.11]). However, there are examples such as conical manifolds (cf. Li [78]) and uniformly elliptic operators (cf. Shen [95] and Caffarelli-Peral [22]) where  $(R_p)$  only holds for  $p$  in a finite interval  $(1, p_0)$ ,  $2 < p_0 < \infty$ . It was discovered in [7] that a natural substitute to  $(GLY_\infty)$  or  $(G_\infty)$  is the  $L^{p_0}$ -boundedness of the gradient of the heat semigroup together with the estimate

$$(G_{p_0}) \quad |||\nabla H_t|||_{p_0 \rightarrow p_0} \leq \frac{C}{\sqrt{t}},$$

$2 < p_0 < \infty$ , which by [7] implies  $(R_p)$  for all  $1 < p < p_0$ . Here and in what follows  $\|\cdot\|_{p \rightarrow p}$  denotes the (sublinear or linear) operator norm from  $L^p(X, \mu)$  to  $L^p(X, \mu)$  for  $p \in [1, \infty]$ . Note conversely that  $(G_p)$  easily implies  $(R_p)$  for any  $p \in (1, \infty)$ .

Observe that  $(G_2)$  always holds. Indeed, it follows from spectral theory that for each  $f \in L^2(X, \mu)$

$$\|\mathcal{L}H_t f\|_2 \leq \frac{C}{t} \|f\|_2,$$

hence

$$|||\nabla H_t f|||_2^2 = \int_X \langle H_t f, \mathcal{L}H_t f \rangle d\mu \leq \frac{C}{t} \|f\|_2^2,$$

i.e.  $(G_2)$ .

By interpolation with  $(G_2)$ ,  $(G_\infty)$  implies  $(G_p)$  for all  $p \in (2, \infty)$ . Finally, if  $(X, d, \mu, \mathcal{E})$  satisfies  $(D)$  and  $(UE)$ , in particular if it satisfies  $(D)$  and  $(P_2)$ , then it follows from the above that  $(GLY_\infty)$  implies  $(G_p)$  for all  $p \in (2, \infty)$ .

Our main results below give a characterisation of  $(G_p)$  for each  $2 < p \leq \infty$  in term of harmonic functions, which can be seen as a gradient version of the equivalence between elliptic and parabolic Harnack inequalities, cf. [58, 15]. Before we state them, let us recall some terminology.

Let  $\Omega \subseteq X$  be a domain. A Sobolev function  $u \in W^{1,2}(\Omega)$  is called a solution to  $\mathcal{L}f = g$  in  $\Omega$  if

$$(1.1) \quad \int_\Omega \langle \nabla f, \nabla \varphi \rangle d\mu = \int_\Omega g(x) \varphi(x) d\mu(x), \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

If  $\mathcal{L}u = 0$  in  $\Omega$ , then we say that  $u$  is harmonic in  $\Omega$ .

**Definition 1.1.** Let  $(X, d, \mu, \mathcal{E})$  be a metric measure Dirichlet space endowed with a “carré du champ” and let  $p \in (2, \infty)$ . We say that the  $L^p$  quantitative reverse Hölder inequality for gradients of harmonic functions holds if  $u$  satisfies  $\mathcal{L}u = 0$  in  $2B$ , then

$$(RH_p) \quad \left( \int_B |\nabla u|^p d\mu \right)^{1/p} \leq \frac{C}{r} \int_{2B} |u| d\mu$$

for some constant  $C$  which does not depend on  $B$  and  $u$ . Analogously,  $(RH_\infty)$  requires that

$$\|\nabla u\|_{L^\infty(B)} \leq \frac{C}{r} \int_{2B} |u| d\mu.$$

In [69, 71],  $(RH_\infty)$  was used to prove isoperimetric inequalities and upper gradient estimates of heat kernels. Note that, under  $(D)$  and  $(P_2)$ ,  $(RH_\infty)$  corresponds to Yau's gradient estimate  $(Y_\infty)$  with  $K = 0$  (see Lemma 2.3 below). See [31, 68, 107, 109] for more about  $(RH_\infty)$ . Actually, the true  $L^p$  reverse Hölder inequality for gradients of harmonic functions,  $(\widetilde{RH}_p)$ , is

$$(\widetilde{RH}_p) \quad \left( \int_B |\nabla u|^p d\mu \right)^{1/p} \leq C \left( \int_{2B} |\nabla u|^2 d\mu \right)^{1/2},$$

if  $u$  is harmonic on  $2B$ ; see [6, 95]. In general,  $(\widetilde{RH}_p)$  is stronger than  $(RH_p)$ . Indeed, as soon as  $(D)$  holds, the Caccioppoli inequality (Lemma 2.4 below) together with a simple covering argument, gives the implication  $(\widetilde{RH}_p) \implies (RH_p)$ . Moreover,  $(RH_p)$  is equivalent to  $(\widetilde{RH}_p)$ , if in addition one has  $(P_2)$ .

Our formulation turns out to be useful, since there exist Riemannian manifolds, where  $(RH_p)$  and  $(G_p)$  holds for some  $p > 2$ , but  $(\widetilde{RH}_p)$  does not hold; see Example 4 from Section 8.

Our first main result gives a characterisation of pointwise estimates of the gradient of the heat kernel.

**Theorem 1.2.** *Let  $(X, d, \mu, \mathcal{E})$  be a non-compact metric measure Dirichlet space endowed with a “carré du champ”. Assume that  $(X, d, \mu, \mathcal{E})$  satisfies  $(D)$ ,  $(UE)$  and  $(P_{\infty, \text{loc}})$ . Then the following statements are equivalent:*

- (i)  $(RH_\infty)$  holds.
- (ii) There exist  $C, c > 0$  such that

$$(GLY_\infty) \quad |\nabla_x h_t(x, y)| \leq \frac{C}{\sqrt{t}V(y, \sqrt{t})} \exp \left\{ -c \frac{d^2(x, y)}{t} \right\}$$

for all  $t > 0$  and a.e.  $x, y \in X$ .

- (iii) The gradient of the heat semigroup  $|\nabla H_t|$  is bounded on  $L^\infty(X)$  for each  $t > 0$  with

$$(G_\infty) \quad \|\nabla H_t\|_{\infty \rightarrow \infty} \leq \frac{C}{\sqrt{t}}.$$

- (iv) There exist  $C, c > 0$  such that

$$(GBE) \quad |\nabla H_t f(x)|^2 \leq C H_{ct}(|\nabla f|^2)(x)$$

for every  $f \in W^{1,2}(X)$ , all  $t > 0$  and a.e.  $x \in X$ .

The new equivalence here is  $(RH_\infty) \iff (GLY_\infty)$ . Indeed, it is easy to see that  $(G_\infty)$  is equivalent to

$$(1.2) \quad \sup_{t>0, x \in M} \sqrt{t} \int_M |\nabla_x p_t(x, y)| d\mu(y) < +\infty,$$



therefore  $(GLY_\infty) \implies (G_\infty)$  follows by integration using  $(D)$  (see [7, p. 919]). Under our assumptions, the upper Gaussian estimate for the heat kernel holds (see [15, Proposition 2.1] for a new proof of this fact). It follows by [7, p.919] again and [34, Theorem 4.11] that  $(G_\infty) \implies (GLY_\infty)$ . The equivalence  $(GLY_\infty) \iff (GBE)$  follows by [7, Lemma 3.3] and [15, Theorem 3.4]. In what follows, we shall call condition  $(iv)$  a generalized Bakry-Émery condition  $(GBE)$ .

Theorem 1.2 admits a direct corollary.

**Corollary 1.3.** *Let  $(X, d, \mu, \mathcal{E})$  be a non-compact metric measure Dirichlet space endowed with a “carré du champ”. Assume that  $(X, d, \mu, \mathcal{E})$  satisfies  $(D)$  and  $(P_2)$ . Then the conditions  $(RH_\infty)$ ,  $(GLY_\infty)$ ,  $(G_\infty)$  and  $(GBE)$  are equivalent.*

In smooth settings, one can drop the assumption  $(P_{\infty, \text{loc}})$  and obtain

**Corollary 1.4.** *Let  $(X, d, \mu)$  be a non-compact Riemannian manifold. Assume that  $(X, d, \mu)$  satisfies  $(D)$  and  $(UE)$ . Then the conditions  $(RH_\infty)$ ,  $(GLY_\infty)$ ,  $(G_\infty)$  and  $(GBE)$  are equivalent.*

**Remark 1.5.** Note that in Theorem 1.2 and Corollary 1.4, we did not require  $(P_2)$  or  $(LY)$ , however, they follow as a consequence of  $(UE)$  together with  $(GLY_\infty)$  or  $(RH_\infty)$ ; cf. [35, 15].

Note that, when  $C = c = 1$ ,  $(GBE)$  is the classical Bakry-Émery condition

$$(BE) \quad |\nabla H_t f(x)|^2 \leq H_t(|\nabla f|^2)(x),$$

which, on manifolds, is known to be equivalent to non-negativity of Ricci curvature; see [11] and also [9, 10, 106]. This equivalence was further generalised to metric measure spaces with non-negative Ricci curvature  $(RCD^*(0, N)$  spaces) by recent works [4, 5, 44].

On Heisenberg groups  $\mathbb{H}(2n, m)$ , according to a recent result by Hu-Li [63] (see Driver and Melcher [37] for earlier results),  $|\nabla H_t f(x)| \leq C H_t(|\nabla f|)(x)$ , and therefore,

$$|\nabla H_t f(x)|^2 \leq C H_t(|\nabla f|^2)(x),$$

since by Jensen  $[H_t(|\nabla f|)]^2 \leq H_t(|\nabla f|^2)$ ; see also [43, 79, 80] for related results. More generally, on sub-Riemannian manifolds satisfying Baudoin-Garofalo’s curvature-dimension inequality  $CD(\rho_1, \rho_2, \kappa, d)$  with  $\rho_1 \geq 0$ ,  $\rho_2 > 0$ ,  $\kappa \geq 0$  and  $d \geq 2$ , it is known that the gradient of the heat kernel satisfies the pointwise inequality  $(GLY_\infty)$  (cf. [12, Theorem 4.2]).

Therefore, by Theorem 1.2, we see that  $(RH_\infty)$ ,  $(GLY_\infty)$  and  $(G_\infty)$  hold on the aforementioned spaces; see the final section for more examples.

As far as  $L^p$  estimates of the gradient of the heat kernel are concerned, we have the following characterisation.

**Theorem 1.6.** *Let  $(X, d, \mu, \mathcal{E})$  be a non-compact metric measure Dirichlet space endowed with a “carré du champ”. Assume that  $(X, d, \mu, \mathcal{E})$  satisfies  $(D)$ ,  $(UE)$  and  $(P_{2, \text{loc}})$ . Let  $p \in (2, \infty)$ . Then the following statements are equivalent:*

(i)  $(RH_p)$  holds.

(ii) There exists  $\gamma > 0$  such that

$$(GLY_p) \quad \int_X |\nabla_x h_t(x, y)|^p \exp \{ \gamma d^2(x, y)/t \} d\mu(x) \leq \frac{C}{t^{p/2} V(y, \sqrt{t})^{p-1}}$$

for all  $t > 0$  and a.e.  $y \in X$ .

(iii) The gradient of the heat semigroup  $|\nabla H_t|$  is bounded on  $L^p(X, \mu)$  for each  $t > 0$  with

$$(G_p) \quad ||| \nabla H_t |||_{p \rightarrow p} \leq \frac{C}{\sqrt{t}}.$$

A Riemannian manifold that is the union of a compact part and a finite number of Euclidean ends is a typical example satisfying  $(UE)$ ,  $(P_{2, \text{loc}})$ , but *not*  $(P_2)$ ; see [24, 33].

Since  $(P_2)$  implies  $(P_{2, \text{loc}})$  and  $(UE)$ , we have the following corollary.

**Corollary 1.7.** *Let  $(X, d, \mu, \mathcal{E})$  be a non-compact metric measure Dirichlet space endowed with a “carré du champ”. Assume that  $(X, d, \mu, \mathcal{E})$  satisfies  $(D)$  and  $(P_2)$ . Let  $p \in (2, \infty)$ . Then the conditions  $(RH_p)$ ,  $(GLY_p)$ , and  $(G_p)$  are equivalent.*

**Remark 1.8.** (i) Note that for  $p = 2$  all conditions (i), (ii), (iii) in Theorem 1.6 hold. This is obvious for (i) and we already observed that this is also the case for (iii). Finally, as far as (ii) is concerned, this follows from [53], see also [33, Lemma 2.3].

(ii) Note also that the limit case  $p = \infty$  of Theorem 1.6 is nothing but Theorem 1.2.

(iii) Theorem 1.6 holds with  $(P_{2, \text{loc}})$  replaced by the weaker condition  $(P_{p, \text{loc}})$ . By [15, Theorem 6.3] together with [16, Corollary 3.8], one can see that,  $(UE)$  and  $(P_{p, \text{loc}})$  together with  $(RH_p)$  or  $(G_p)$  imply  $(P_{2, \text{loc}})$ .

(iv) Finally, note that under  $(D)$  and  $(P_2)$ , there always exists  $\varepsilon > 0$  such that  $(RH_p)$ , hence  $(GLY_p)$  and  $(G_p)$ , hold for  $2 < p < 2 + \varepsilon$ ; see [6, Section 2.1] and Lemma 6.2 below.

To the best of our knowledge, Theorem 1.2, Theorem 1.6 and their local versions, Theorem 5.2, Theorem 5.3, are new even on Riemannian manifolds. Since our assumptions are quite mild, our setting includes Riemannian metric measure spaces, sub-Riemannian manifolds, and degenerate elliptic/parabolic equations in these settings; see the final section.

Regarding the proofs, the main difficulties and novelties appear in the proof of  $(RH_p) \implies (GLY_p)$  for  $p \in (2, \infty)$ , and in  $(G_p) \implies (RH_p)$  for  $p \in (2, \infty]$ .

A version of the implication  $(RH_\infty) \implies (GLY_\infty)$  was proven in [69, Theorem 3.2] via the quantitative regularity estimates for solutions to the Poisson equation in [71, Theorem 3.1], under  $(D)$  and  $(P_2)$ . In the present work, we replace the assumptions  $(D)$  and  $(P_2)$  there by the slight weaker combinations  $(D)$ ,  $(UE)$  and  $(P_{\infty, \text{loc}})$ . To prove  $(RH_p) \implies (GLY_p)$  for  $p \in (2, \infty)$ , we follow some ideas from [69, 71]. In particular, starting from  $(RH_p)$ , we first establish a quantitative regularity estimate for solutions to the Poisson equation; see Theorem 3.5 below. Recall that harmonic functions are *not* necessarily locally Lipschitz in a non-smooth setting. Therefore, to establish Theorem 3.5, we can neither assume nor use any Lipschitz regularity of harmonic functions. In the classical setting, the fact that quantitative regularity for harmonic functions implies

quantitative regularity for solutions to the Poisson equation is easy to prove and there is even an analog for certain non-linear equations, see [77].

To overcome the difficulties attached to the non-smooth setting, we use a pointwise characterisation of Sobolev spaces on metric measure spaces discovered by Hajlasz [56]; see [59, 94] and Section 2.1 below for more details. Then by using  $(RH_p)$  in the full strength, a stopping time argument and a bootstrap argument, we obtain pointwise control on Hajlasz gradients of solutions to the Poisson equation in terms of potentials; see (3.4) below. We expect that such estimates have independent interest.

Then, by viewing the heat kernel  $h_t$  as a solution to the Poisson equation  $\mathcal{L}h_t = -\frac{\partial h_t}{\partial t}$ , where the estimate of  $\frac{\partial h_t}{\partial t}$  can be obtained from  $h_t$  by using Cauchy transforms (cf. Sturm [100, Theorem 2.6]), we obtain  $(GLY_p)$ .

To prove  $(G_p) \implies (RH_p)$  for  $p \in (2, \infty]$ , we first establish a reproducing formula for harmonic functions by using the finite propagation speed property; see Lemma 4.6 below. Then, by using this reproducing formula, we follow recent developments on mapping properties of the boundedness of spectral multipliers from [15, 21] to show that  $(G_p) \implies (RH_p)$  for all  $p \in (2, \infty]$ .

However as we want to prove that this implication is valid for the local version of our assumption (see Section 5), we will provide also a more self-contained but more complicated argument, which is based on:

- (i) spectral theory;
- (ii) Sobolev embedding, which has a local version under a local doubling condition together with a local upper Gaussian estimate of the heat kernel;
- (iii)  $L^p$ -boundedness of  $|\nabla H_t|$ , i.e.,  $(G_p)$ , which also has a local version  $(G_{p,\text{loc}})$ ; see Section 5.

Starting from (i), by using (ii) and (iii), and an additional bootstrap argument in the case where  $p$  is large, we can show again that  $(G_p) \implies (RH_p)$ ; see the second proof of Theorem 4.9 below.

## 1.2 Applications to Riesz transforms

Let us apply the previous characterisations to the Riesz transform  $|\nabla \mathcal{L}^{-1/2}|$ . The negative square roots of the non-negative self-adjoint operator  $\mathcal{L}$  can be computed as

$$\mathcal{L}^{-1/2} = \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-s\mathcal{L}} \frac{ds}{\sqrt{s}}.$$

We refer the reader to [7] for more details. The Riesz transform is the sublinear operator  $|\nabla \mathcal{L}^{-1/2}|$ . We say that  $(R_p)$  holds if this operator is continuous from  $L^p(X, \mu)$  to itself. One checks easily that  $(R_2)$  follows from the definitions and spectral theory.

For  $p \in (1, 2)$ , it was proved by Coulhon and Duong in [33] that  $(R_p)$  holds as soon as  $(D)$  and  $(UE)$  hold (however, this condition is not necessary, see [29]). In particular,  $(D)$  and  $(P_2)$  are sufficient conditions for  $(R_p)$  to be valid in this range.

For  $p > 2$ , Auscher, Coulhon, Duong and Hofmann established in [7] a remarkable characterisation of the boundedness of the Riesz transform via boundedness of the gradient of the heat semi-group on manifolds. Although the characterisation in [7] was stated on manifolds, its proof indeed

works on metric measure spaces, as indicated in [15, p.6]. We refer the reader to [6, 8, 12, 15, 16] for more on this aspect.

Using [7, Theorem 1.3], Theorem 1.6 above, and the open-ended character of condition  $(RH_p)$  (Lemma 6.2 below), we obtain the following result.

**Theorem 1.9.** *Assume that the non-compact metric measure Dirichlet space  $(X, d, \mu, \mathcal{E})$  satisfies  $(D)$  and  $(UE)$ . Let  $p \in (2, \infty)$ . If  $(P_p)$  holds, then  $(RH_p)$ ,  $(G_p)$  and  $(R_p)$  are equivalent.*

Let us compare Theorem 1.6 and Theorem 1.9 with [6, Theorem 2.1]. The latter result states that on a Riemannian manifold  $M$  satisfying  $(D)$  and  $(P_2)$ , there exists  $p_M \in (2, \infty]$  such that for all  $p_0 \in (2, p_M)$ ,  $(RH_p)$  for all  $p \in (2, p_0)$  is equivalent to the validity of  $(R_p)$  for all  $p \in (2, p_0)$ . Now  $(R_p)$  easily implies  $(G_p)$  and conversely, according to [7, Theorem 2.1], under the same assumptions the validity of  $(G_p)$  for all  $p \in (2, p_0)$  implies the validity of  $(R_p)$  for all  $p \in (2, p_0)$ . Theorem 1.6 and Theorem 1.9 contain three improvements with respect to [6, Theorem 2.1]. First, the proof of [6, Theorem 2.1] makes an essential use of 1-forms on manifolds, and we do not know how to extend the arguments from [6] to our general setting. Second, Theorem 1.6 and Theorem 1.9 state a point-to-point equivalence among  $(RH_p)$ ,  $(G_p)$  and  $(R_p)$ , as opposed to a mere equivalence between  $(RH_p)$  for  $p \in (2, p_0)$  and  $(G_p)$  for  $p \in (2, p_0)$ . Finally, we obtain that  $p_M = +\infty$ .

According to the Gehring's Lemma (cf. [51, 65]), our reverse Hölder inequality  $(RH_p)$  is an open-ended condition; see Lemma 6.2 below. We then have the following corollary of Theorem 1.9, which generalizes the main result of [6] and a recent result [16, Theorem 1.2].

**Corollary 1.10.** *Assume that the non-compact metric measure Dirichlet space  $(X, d, \mu, \mathcal{E})$  satisfies  $(D)$  and  $(UE)$ .*

- (i) *If  $(P_2)$  holds, then the set of  $p$ 's such that  $(R_p)$  holds is an interval  $(1, p_0)$ , with  $p_0 \in (2, \infty]$ .*
- (ii) *Let  $p \in (2, \infty)$ . If  $(P_p)$ , and one of the conditions  $(RH_p)$ ,  $(G_p)$ ,  $(R_p)$ , hold, then there exists  $\varepsilon > 0$  such that  $(R_{p+\varepsilon})$  holds.*

**Remark 1.11.** In Theorem 1.9 and (ii) of Corollary 1.10, even though we only assumes  $(P_p)$ , but recently results from [15, Theorem 6.3] and [16, Corollary 3.8] show that  $(P_p)$  together with  $(RH_p)$  or  $(G_p)$  implies  $(P_2)$ .

### 1.3 Sobolev inequality and isoperimetric inequality

We say that the Sobolev inequality  $(S_{p,q})$  holds if for each Lipschitz function  $f$ , compactly supported in any ball  $B$ ,  $B = B(x_0, r)$ ,

$$(S_{q,p}) \quad \left( \int_B |f|^q d\mu \right)^{1/q} \leq Cr \left( \int_B |\nabla f|^p d\mu \right)^{1/p},$$

where  $C = C(p, q)$ .

Applying the methods from [70, 71], we will show that on a metric measure space  $(X, d, \mu, \mathcal{E})$  satisfying  $(D_Q)$  and  $(UE)$ , if additionally for some  $p_0 \in (2, \infty)$ ,  $(P_{p_0, \text{loc}})$  and one of the conditions  $(RH_{p_0})$ ,  $(R_{p_0})$ ,  $(G_{p_0})$  holds, then the Sobolev inequality  $(S_{q, p'_0})$ , where  $p'_0 < 2$  is the Hölder

conjugate of  $p_0$ ,  $q \geq p'_0$  satisfying  $1/p'_0 - 1/q < 1/Q$ , is valid. Analogues for the isoperimetric inequality ( $p_0 = \infty$ ) can also be established.

## 1.4 Plan of the paper

The paper is organized as follows. In Section 2, we recall and provide some basic notions and tools, which include Sobolev spaces, harmonic functions, Poisson equations and some functional calculi.

In Section 3, we provide a quantitative gradient estimate for solutions to the Poisson equations, assuming  $(RH_p)$ .

In Section 4, we give the proofs of Theorem 1.2 and Theorem 1.6, and their corollaries.

In Section 5, we provide the local versions of our main results, Theorem 5.2 and Theorem 5.3, which also cover compact cases.

In Section 6, we prove Theorem 1.9, and in Section 7, we study Sobolev inequalities and the isoperimetric inequality.

In Section 8, we exhibit several examples that our results can be applied to.

Throughout the work, we denote by  $C, c$  positive constants which are independent of the main parameters, but which may vary from line to line. We use  $\sim$  to mean that two quantities are comparable.

## 2 Preliminaries and auxiliary tools

### 2.1 Harmonic functions and Poisson equations

In this subsection, we recall some basic properties of harmonic functions and of solutions to the Poisson equation. Most of these properties were deduced via the de Giorgi-Moser-Nash theory, requiring only (local) doubling property and (local) Sobolev inequality.

Notice that  $(UE)$  is equivalent to the Sobolev inequality  $(LS_q)$ , for any  $q \in (2, \infty]$  satisfying  $\frac{q-2}{q} < \frac{2}{Q}$ . Under  $(LS_q)$ , it holds for each Lipschitz function with compact support in  $B$ , and by density for each  $f \in W_0^{1,2}(B)$ , that

$$(S_{q,2}) \quad \left( \int_B |f|^q d\mu \right)^{1/q} \leq C_S r \left( \int_B |\nabla f|^2 d\mu \right)^{1/2}.$$

see [21, 57].

We continue with the Harnack inequality; see [18, 19, 67] for instance.

**Proposition 2.1.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies  $(D)$  and  $(UE)$ . Then there exists  $C$  depends only on  $C_D$  and  $C_S$  such that if  $\mathcal{L}u = 0$  in  $B(x_0, r)$ ,*

$$\|u\|_{L^\infty(B(x_0, r/2))} \leq C \int_{B(x_0, r)} |u| d\mu.$$

**Proposition 2.2.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies (D) and  $(P_{2, \text{loc}})$  hold. For each  $r_0 > 0$ , there exists  $C = C(C_D, C_P(r_0))$  such that if  $u$  is a positive harmonic function on  $B(x_0, r)$ ,  $r < r_0$ , then*

$$\sup_{y \in B(x_0, r/2)} u(y) \leq C \inf_{y \in B(x_0, r/2)} u(y).$$

*Further if  $(P_2)$  holds, then the above constant  $C$  can be independent of  $r_0$ .*

Using the Harnack inequality, we obtain the following relation between Yau's gradient estimate and our condition  $(RH_\infty)$ .

**Lemma 2.3.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies (D) and  $(P_2)$ . Then  $(RH_\infty)$  holds if and only if,  $(Y_\infty)$  holds with  $K = 0$ .*

*Proof.*  $(RH_\infty) \implies (Y_\infty)$  with  $K = 0$ : Suppose that  $u$  is positive harmonic function on  $2B$ ,  $B = B(x_0, r)$ . By using Proposition 2.1 and Proposition 2.2 and a simple covering argument, we see that

$$|\nabla u(x)| \leq \|\nabla u\|_{L^\infty(B)} \leq \frac{C}{r} \int_{\frac{3}{2}B} |u| d\mu \leq \frac{C}{r} \sup_{y \in \frac{3}{2}B} u(y) \leq \frac{C}{r} \inf_{y \in \frac{3}{2}B} u(y) \leq \frac{C}{r} u(x)$$

for a.e.  $x \in B$ , i.e.,  $(Y_\infty)$  holds with  $K = 0$ .

$(Y_\infty)$  with  $K = 0 \implies (RH_\infty)$ : Suppose that  $u$  is a harmonic function in  $2B$ . Let  $\delta = \|u\|_{L^\infty(\frac{3}{2}B)}$ , then the strong maximum principle (cf. [19]) implies that either  $u + \delta \equiv 0$  in  $\frac{3}{2}B$  or  $u + \delta > 0$  there. In the first case,  $(RH_\infty)$  holds obviously since  $|\nabla u| \equiv 0$  in  $B$ . For the second case, by using Proposition 2.1 and a covering argument again, we obtain

$$\delta = \|u\|_{L^\infty(\frac{3}{2}B)} \leq C \int_{2B} |u| d\mu,$$

and hence by  $(Y_\infty)$  with  $K = 0$ ,

$$|\nabla(u(x) + \delta)| \leq \frac{C}{r} [u(x) + \delta] \leq \frac{C}{r} \inf_{y \in B} [u(y) + \delta] \leq \frac{C}{r} \int_{2B} |u + \delta| d\mu \leq \frac{C}{r} \int_{2B} |u| d\mu$$

for a.e.  $x \in B$ . That is,  $(RH_\infty)$  holds, which completes the proof.  $\square$

We need the following Caccioppoli inequality; see [19, 67].

**Lemma 2.4.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies (D). Then if  $\mathcal{L}f = g$  in  $B := B(x_0, r)$ ,  $g \in L^2(B)$ ,*

$$\int_{B(x_0, r)} |\nabla f|^2 d\mu \leq \frac{C}{(R-r)^2} \int_{B(x_0, R)} |f|^2 d\mu + C(R-r)^2 \int_{B(x_0, R)} |g|^2 d\mu,$$

where  $C$  is an absolute constant, independent of  $C_D, C_P$ .

**Lemma 2.5** ([19]). *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies  $(D_Q)$ ,  $Q > 1$ , and  $(UE)$ . Let  $p \in (\frac{Q}{2}, \infty] \cap (1, \infty]$ . Then for each  $g \in L^p(B(y_0, r))$ , there is a unique solution  $f \in W_0^{1,2}(B(y_0, r))$  to  $\mathcal{L}f = g$  in  $B(y_0, r)$ . Moreover*

$$\|f\|_{L^\infty(B(y_0, r))} \leq Cr^2 V(y_0, r)^{-1/p} \|g\|_{L^p(B(y_0, r))},$$

where  $C = C(C_D, C_S)$ .

*Proof.* See [19, Theorem 4.1] for the existence and the given estimate; the uniqueness follows since the difference of any two solutions is harmonic, with boundary value zero in the Sobolev sense.  $\square$

**Definition 2.6.** *In what follows, we will always let  $p_Q := 1$  when  $Q \in (1, 2)$ , choose  $p_Q \in (1, 2)$  that can be taken arbitrarily close to  $\frac{2Q}{Q+2}$  when  $Q \geq 2$ .*

**Remark 2.7.** If one has  $(P_2)$ , then the Sobolev inequality  $(S_{q,2})$  holds with  $q = \frac{2Q}{Q-2}$  when  $Q > 2$ . In this case, one can take in the above definition  $p_Q = \frac{2Q}{Q+2}$  for  $Q > 2$ .

**Lemma 2.8.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies  $(D_Q)$ ,  $Q > 1$ , and  $(UE)$ . For each  $g \in L^q(B)$ ,  $B = B(x_0, r)$ ,  $q \geq p_Q$ , there exists  $f \in W_0^{1,2}(B)$  that satisfies  $\mathcal{L}f = g$  in  $B$ . Moreover, there exists  $C = C(C_D, C_S)$  such that*

$$\int_B |f| d\mu \leq Cr \left( \int_B |\nabla f|^2 d\mu \right)^{1/2} \leq Cr^2 \left( \int_B |g|^{p_Q} d\mu \right)^{1/p_Q}.$$

*Proof.* Let us first prove the existence. If  $q \in (\frac{Q}{2}, \infty] \cap (2, \infty]$ , then the existence follows from Lemma 2.5. Assume now that  $p_Q \leq q \leq \max\{Q/2, 2\}$ . Notice that  $(S_{q,2})$  holds with  $q$  be the Hölder conjugate of  $p_Q$ . For each  $k \in \mathbb{N}$ , let  $g_k := g\chi_{|g| \leq k}$ . By Lemma 2.5, there exists a solution  $f_k \in W_0^{1,2}(B)$  to  $\mathcal{L}f_k = g_k$  in  $B$ . For all  $k, j \in \mathbb{N}$ , the Sobolev inequality  $(S_{(p_Q)', 2})$  gives

$$\begin{aligned} \int_B |\nabla(f_k - f_j)|^2 d\mu &= \int_B (f_k g_k + f_j g_j - f_k g_j - f_j g_k) d\mu \\ &= \int_B [g_k - g_j][f_k - f_j] d\mu \\ &\leq \|g_k - g_j\|_{L^{p_Q}(B)} \|f_k - f_j\|_{L^{p'_Q}(B)} \\ &\leq C \|g_k - g_j\|_{L^{p_Q}(B)} \frac{r}{\mu(B)^{1/2-1/p'_Q}} \|\nabla(f_k - f_j)\|_{L^2(B)}, \end{aligned}$$

and similarly

$$(2.1) \quad \int_B |\nabla f_k|^2 d\mu \leq C \|g_k\|_{L^{p_Q}(B)} \frac{r}{\mu(B)^{1/2-1/p'_Q}} \|\nabla f_k\|_{L^2(B)},$$



where we used the fact that  $q \in (p_Q, \max\{Q/2, 2\}]$ . Therefore,  $(f_k)_k$  is a Cauchy sequence in  $W_0^{1,2}(B)$ , and there exists a limit  $f \in W_0^{1,2}(B)$ . By this and the Sobolev inequality  $(S_{(p_Q)', 2})$ , we see that for each  $\varphi \in W_0^{1,2}(B)$ ,

$$\int_B \langle \nabla f, \nabla \varphi \rangle d\mu = \lim_{k \rightarrow \infty} \int_B \langle \nabla f_k, \nabla \varphi \rangle d\mu = \lim_{k \rightarrow \infty} \int_B g_k \varphi d\mu = \int_B g \varphi d\mu,$$

where the last equality follows from the convergence  $g_k \rightarrow g$  in  $L^{p_Q}(B)$  together with  $\varphi \in W_0^{1,2}(B) \subset L^{p_Q'}(B)$ . This implies that  $f$  is a solution to  $\mathcal{L}f = g$  in  $B$ .

Notice that by (2.1),

$$\int_B |f_k|^2 d\mu \leq Cr^2 \int_B |\nabla f_k|^2 d\mu \leq C \|g_k\|_{L^{p_Q}(B)}^2 \frac{r^4}{\mu(B)^{2/p_Q}} \leq C \|g\|_{L^{p_Q}(B)}^2 \frac{r^4}{\mu(B)^{2/p_Q}}.$$

By this, letting  $k \rightarrow \infty$ , we can conclude that

$$\int_B |f| d\mu \leq Cr \left( \int_B |\nabla f|^2 d\mu \right)^{1/2} \leq Cr^2 \left( \int_B |g|^{p_Q} d\mu \right)^{1/p_Q},$$

as desired.  $\square$

**Lemma 2.9.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies  $(D_Q)$ ,  $Q > 1$ , and  $(P_{2, \text{loc}})$ . Suppose that  $f \in W^{1,2}(B)$ ,  $B = B(x_0, r)$ ,  $g \in L^p(B)$  and  $\mathcal{L}f = g$  in  $B$ ,  $p \in (\frac{Q}{2}, \infty] \cap (2, \infty]$ . Then  $f$  is locally Hölder continuous on  $B$ .*

## 2.2 Functional calculus

Let  $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ . Let  $L$  be a non-negative, self-adjoint operator on  $L^2(X, \mu)$ , we denote by  $E_L(\lambda)$  its spectral decomposition. Then, for every bounded measurable function  $F : [0, \infty) \rightarrow \mathbb{C}$ , one defines the operator  $F(L) : L^2(X, \mu) \rightarrow L^2(X, \mu)$  by the formula

$$(2.2) \quad F(L) := \int_0^\infty F(\lambda) dE_L(\lambda).$$

In the case of  $F_z(\lambda) := e^{-z\lambda}$  for  $z \in \mathbb{C}_+$ , one sets  $e^{-zL} := F_z(L)$  as given by (2.2), which gives a definition of the heat semigroup for complex time. By spectral theory, the family  $\{e^{-zL}\}_{z \in \mathbb{C}_+}$  satisfies

$$\|e^{-zL}\|_{2 \rightarrow 2} \leq 1$$

for all  $z \in \mathbb{C}_+$ .

**Definition 2.10** (Davies-Gaffney estimate). *We say that the semigroup  $\{e^{-tL}\}_{t>0}$  satisfies the Davies-Gaffney estimate if for all open sets  $E$  and  $F$  in  $X$ ,  $t \in (0, \infty)$  and  $f \in L^2(E)$  ( $\operatorname{supp} f \subset E$ ),*

$$(2.3) \quad \|e^{-tL} f\|_{L^2(F)} \leq \exp \left\{ - \frac{\operatorname{dist}(E, F)^2}{4t} \right\} \|f\|_{L^2(E)},$$

where and in what follows,  $\operatorname{dist}(E, F) := \inf_{x \in E, y \in F} d(x, y)$ .



**Definition 2.11** (Finite propagation speed property). *We say that  $L$  satisfies the finite propagation speed property if for all  $0 < t < d(E, F)$  and  $E, F \subset X$ ,  $f_1 \in L^2(E)$  and  $f_2 \in L^2(F)$ ,*

$$(2.4) \quad \int_X \langle \cos(t \sqrt{L}) f_1, f_2 \rangle d\mu = 0.$$

The following result was obtained by Sikora in [96]. The statement can be also found in [61, Proposition 3.4] and [34, Theorem 3.4].

**Proposition 2.12.** *The operator  $L$  satisfies the finite propagation speed property (2.4) if and only if the semigroup  $\{e^{-tL}\}_{t>0}$  satisfies the Davies-Gaffney estimate (2.3).*

By the Fourier inversion formula, whenever  $F$  is an even bounded Borel-function with  $\hat{F} \in L^1(\mathbb{R})$ , we can write  $F(\sqrt{L})$  in terms of  $\cos(t \sqrt{L})$ , as

$$(2.5) \quad F(\sqrt{L}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(t) \cos(t \sqrt{L}) dt.$$

The following result follows from [100, Theorem 0.1] (see also [60]) and [34, Theorem 3.4].

**Lemma 2.13.** *Let  $(X, d, \mu, \mathcal{E})$  be a metric measure Dirichlet space endowed with a “carré du champ”. Then the associated heat semigroup  $e^{-t\mathcal{L}}$  satisfies the Davies-Gaffney estimate.*

In what follows,  $\mathcal{L}$  is as above. Let  $\mathcal{S}(\mathbb{R})$  denote the collection of all Schwartz functions on  $\mathbb{R}$ . We need the following  $L^2$ -boundedness of spectral multipliers.

**Lemma 2.14.** *Let  $\Phi \in \mathcal{S}(\mathbb{R})$  be an even function with  $\Phi(0) = 1$ . Then there exists  $C > 0$  such that*

$$\sup_{r>0} \|(r^2 \mathcal{L})^{-1} (1 - \Phi(r \sqrt{\mathcal{L}}))\|_{2 \rightarrow 2} \leq C,$$

and, for each  $k = 0, 1, 2, \dots$ , there exists  $C$  such that

$$\sup_{r>0} \|(r^2 \mathcal{L})^k \Phi(r \sqrt{\mathcal{L}})\|_{2 \rightarrow 2} \leq C.$$

*Proof.* We only give the proof of the first inequality; the second one follows similarly. Notice that  $\Phi'(0) = 0$  so by spectral theory

$$\|(r^2 \mathcal{L})^{-1} (1 - \Phi(r \sqrt{\mathcal{L}}))\|_{2 \rightarrow 2} \leq \sup_{\lambda} \left| \frac{1 - \Phi(r\lambda)}{r^2 \lambda^2} \right| = \sup_{\lambda} \left| \frac{1 - \Phi(\lambda)}{\lambda^2} \right| < \infty.$$

The proof is complete.  $\square$

**Lemma 2.15.** *Let  $\Phi \in \mathcal{S}(\mathbb{R})$  be an even function whose Fourier transform  $\hat{\Phi}$  satisfies  $\text{supp } \hat{\Phi} \subset [-1, 1]$ . Then for every  $\kappa \in \mathbb{Z}_+$  and  $t > 0$ , the operator  $(t^2 \mathcal{L})^\kappa \Phi(t \sqrt{\mathcal{L}})$  satisfies*

$$(2.6) \quad \int_X \langle (t^2 \mathcal{L})^\kappa \Phi(t \sqrt{\mathcal{L}}) f_1, f_2 \rangle d\mu = 0$$

for all  $0 < t < d(E, F)$  and  $E, F \subset X$ ,  $f_1 \in L^2(E)$  and  $f_2 \in L^2(F)$ .

*Proof.* Let  $\Phi_\kappa(s) := s^{2\kappa}\Phi(s)$ . By noticing that  $\widehat{\Phi_\kappa}(\lambda) = (-1)^\kappa \frac{d^{2\kappa}}{d\lambda^{2\kappa}} \widehat{\Phi}(\lambda)$ , the conclusion follows from Lemma 2.13, Proposition 2.12 and (2.5).  $\square$

**Lemma 2.16.** *Let  $\Phi \in \mathcal{S}(\mathbb{R})$  be an even function with  $\Phi(0) = 1$ . Then, for each  $f \in L^2(X, \mu)$ , it holds that*

$$\lim_{t \rightarrow 0^+} \|f - \Phi(t\sqrt{\mathcal{L}})f\|_2 = 0.$$

*Proof.* The domain  $\mathcal{D}(\mathcal{L})$  is dense in  $L^2(X, \mu)$  so it is enough to prove Lemma 2.16 for  $f \in \mathcal{D}(\mathcal{L})$ . Then

$$\|f - \Phi(t\sqrt{\mathcal{L}})f\|_2 \leq Ct^2 \|\mathcal{L}f\|_2 \left\| (t^2 \mathcal{L})^{-1} (1 - \Phi(t\sqrt{\mathcal{L}})) \right\|_{2 \rightarrow 2}$$

and the lemma follows from Lemma 2.14.  $\square$

### 3 Regularity of solutions to the Poisson equation

#### 3.1 A priori estimates for solutions to the Poisson equation

In this subsection, we show that regularity of harmonic functions implies a gradient estimate for solutions to the Poisson equation  $\mathcal{L}f = g$ , under the assumption that  $g$  is bounded.

The following result was established in [71, Proposition 3.1] under the stronger condition  $(D_Q)$  and  $(P_2)$ , we report a proof for completeness. Given  $a > 1$  and  $r > 0$ , let  $[\log_a r]$  be the largest integer smaller than  $\log_a r$ .

**Proposition 3.1.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies  $(D_Q)$ ,  $Q > 1$ , and  $(UE)$ . Suppose that  $\mathcal{L}f = g$  in  $2B$ ,  $B = B(x_0, r)$ , with  $g \in L^\infty(2B)$ . Then there exists  $C > 0$  such that for almost every  $x \in B$ ,*

$$|f(x)| \leq C \left\{ \int_{2B} |f| d\mu + G_1(x) \right\},$$

where

$$(3.1) \quad G_1(x) := \sum_{j \leq [\log_2 r] + 2} 2^{2j} \left( \int_{B(x, 2^j)} |g|^{p_Q} d\mu \right)^{1/p_Q},$$

where  $p_Q$  is as in Definition 2.6.

*Proof.* Let  $k_0 = [\log_2 r] + 2$  and  $x \in B$ . By Lemma 2.8 that for each  $j \leq k_0 + 2$ , there exists  $f_j \in W_0^{1,2}(B(x, 2^j))$  such that  $\mathcal{L}f_j = g$  in  $B(x, 2^j)$ , and

$$\int_{B(x, 2^{j-2})} |f_j(y)| d\mu(y) \leq C \int_{B(x, 2^j)} |f_j(y)| d\mu(y) \leq C 2^{2j} \left( \int_{B(x, 2^j)} |g|^{p_Q} d\mu \right)^{1/p_Q}.$$

Moreover, for each  $k \leq k_0 - 1$ , as  $\mathcal{L}(f_{k+1} - f_k) = 0$  in  $B(x, 2^k) \cap 2B$ , by Proposition 2.1, we have

$$\|f_{k+1} - f_k\|_{L^\infty(B(x, 2^{k-2}))} \leq C \int_{B(x, 2^{k-1})} |f_{k+1} - f_k| d\mu \leq C \int_{B(x, 2^k)} |f_{k+1} - f_k| d\mu.$$

Thus, from the above two inequalities, for almost every  $x \in B$ , we deduce that

$$\begin{aligned} |f(x)| &= \lim_{j \rightarrow -\infty} \int_{B(x, 2^{j-2})} |f(y)| d\mu(y) \\ &\leq \limsup_{j \rightarrow -\infty} \left\{ \int_{B(x, 2^{j-2})} |f_j(y)| d\mu(y) + \sum_{k=j}^{k_0-1} \int_{B(x, 2^{j-2})} |f_{k+1}(y) - f_k(y)| d\mu(y) \right\} \\ &\quad + \limsup_{j \rightarrow -\infty} \int_{B(x, 2^{j-2})} |f_{k_0}(y) - f(y)| d\mu(y) \\ &\leq \limsup_{j \rightarrow -\infty} \left\{ C 2^{2j} \left( \int_{B(x, 2^j)} |g|^{p_Q} d\mu \right)^{1/p_Q} + \sum_{k=j}^{k_0-1} \|f_{k+1} - f_k\|_{L^\infty(B(x, 2^{j-2}))} + \|f_{k_0} - f\|_{L^\infty(B(x, 2^{j-2}))} \right\} \\ &\leq \lim_{j \rightarrow -\infty} C \sum_{k=j}^{k_0-1} \int_{B(x, 2^k)} |f_{k+1}(y) - f_k(y)| d\mu(y) + C \int_{B(x, 2^{k_0-3})} |f_{k_0} - f| d\mu \\ &\leq C \sum_{k=-\infty}^{k_0-3} \int_{B(x, 2^k)} |f_{k+1}(y)| d\mu(y) + C \int_{B(x, 2^{k_0-3})} |f_{k_0}| d\mu + C \int_{B(x, 2^{k_0-3})} |f| d\mu \\ &\leq C \sum_{k=-\infty}^{k_0} 2^{2k} \left( \int_{B(x, 2^k)} |g|^{p_Q} d\mu \right)^{1/p_Q} + C \int_{2B} |f| d\mu. \end{aligned}$$

Above, in the third inequality, we used the fact that

$$\limsup_{j \rightarrow -\infty} 2^{2j} \left( \int_{B(x, 2^j)} |g|^{p_Q} d\mu \right)^{1/p_Q} \leq \limsup_{j \rightarrow -\infty} 2^{2j} \|g\|_{L^\infty(2B)} = 0,$$

and in the last inequality, we used the doubling condition to conclude that

$$\int_{B(x, 2^{k_0-1})} |f| d\mu \leq C \int_{2B} |f| d\mu.$$

The proof is complete.  $\square$

Above, in case  $B(x, 2^j)$  is not entirely contained in  $2B$ , the integral of  $|g|$  over  $B(x, 2^j)$  refers to the integral of the zero extension of  $g$  to the exterior of  $2B$ . For what follows, we fix  $p_Q$  as in Definition 2.6.

The next statement deals with the case when harmonic functions are Lipschitz; the proof is similar to that of [71, Theorem 3.1].

**Theorem 3.2.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies  $(D_Q)$ ,  $Q > 1$ ,  $(UE)$  and  $(P_{\infty, \text{loc}})$ . Assume that  $(RH_{\infty})$  holds. Then if  $\mathcal{L}f = g$  in  $2B$ ,  $B = B(x_0, r)$ ,  $g \in L^{\infty}(2B)$ , there exists  $C = C(C_D, C_{LS}, C_P(1)) > 0$  such that, for almost every  $x \in B$ ,*

$$|\nabla f(x)| \leq C \left\{ \frac{1}{r} \int_{2B} |f| d\mu + G_2(x) \right\},$$

where

$$G_2(x) := \sum_{j \leq [\log_2 r] + 2} 2^j \left( \int_{B(x, 2^j)} |g|^{p_Q} d\mu \right)^{1/p_Q}.$$

For its proof, we need the following Lipschitz regularity, which follows from  $(D)$ ,  $(P_{\infty, \text{loc}})$  and  $(RH_{\infty})$ . Its proof uses a telescopic estimate, which will be omitted; see for instance [94].

**Lemma 3.3.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies  $(D)$ ,  $(UE)$  and  $(P_{\infty, \text{loc}})$ . Assume that  $(RH_{\infty})$  holds. If  $\mathcal{L}f = 0$  in  $2B$ ,  $B = B(x_0, r)$ , then for almost all  $x, y \in B(x_0, r)$  with  $d(x, y) < 1/2$ , it holds that*

$$|f(x) - f(y)| \leq C \frac{d(x, y)}{r} \int_{2B} |f| d\mu,$$

where  $C = C(C_D, C_P(1))$ .

*Proof of Theorem 3.2.* Set  $k_0 := [\log_2 r] + 2$ . Let  $x, y \in B$  be the Lebesgue points of  $f$ . For  $d(x, y) \geq r/16$ , by Proposition 3.1, we have

$$(3.2) \quad |f(x) - f(y)| \leq C \int_{2B} |f| d\mu + CG_1(x) + G_2(y) \leq Cd(x, y) \left\{ \frac{1}{r} \int_{2B} |f| d\mu + G_2(x) + G_2(y) \right\}.$$

Now assume that  $d(x, y) < r/16$  and  $d(x, y) < \frac{1}{2}$ . Choose  $k_1 \in \mathbb{Z}$  such that  $2^{k_1} \leq d(x, y) < 2^{k_1+1}$ .

As in the proof of Proposition 3.1, for each  $j \in \mathbb{Z}$  and  $j \leq k_0$ , let  $f_j \in W_0^{1,2}(B(x, 2^j))$  such that  $\mathcal{L}f_j = g$  in  $B(x, 2^j)$ . By the choice of  $k_1$ , we see that for each  $z \in B(y, 2^{k_1+1})$ ,

$$d(x, z) \leq d(x, y) + d(y, z) < 2^{k_1+1} + 2^{k_1+1} \leq 2^{k_1+2},$$

which further implies that  $B(y, 2^{k_1+1}) \subset B(x, 2^k)$  for each  $k \geq k_1 + 2$ . Hence for each  $k \geq k_1 + 2$ , the value  $f_k(y)$  is well defined, and we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_{k_0}(x) - [f(y) - f_{k_0}(y)]| \\ &\quad + \sum_{j=k_1+2}^{k_0-1} |[f_j(x) - f_{j+1}(x)] - [f_j(y) - f_{j+1}(y)]| + |f_{k_1+2}(x) - f_{k_1+2}(y)| \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

Let us estimate the term  $I_1$ . According to the choice  $f - f_{k_0}$  is harmonic in  $B(x, 2^{k_0}) \cap 2B$ . By using the fact  $y \in B(x, 2^{k_1+1})$  together with Lemma 3.3, we conclude that

$$I_1 \leq C \frac{d(x, y)}{2^{k_0}} \int_{B(x, 2^{k_0})} |f - f_{k_0}| d\mu$$

$$\leq Cd(x, y) \left\{ \frac{1}{r} \int_{2B} |f| d\mu + 2^{k_0} \left( \int_{B(x, 2^{k_0})} |g|^{p_Q} d\mu \right)^{1/p_Q} \right\},$$

where we used (D),  $\frac{1}{\mu(B(x, 2^{k_0}))} \leq C \frac{1}{\mu(2B)}$ , and Lemma 2.8 to estimate  $\int_{B(x, 2^{k_0})} |f_{k_0}| d\mu$ .

The term  $I_2$  can be estimated similarly to the first term. For each  $k_1 + 2 \leq j \leq k_0 - 1$ ,  $f_j - f_{j+1}$  is harmonic in  $B(x, 2^j)$ . As  $y \in B(x, 2^{k_1+1}) \subset \frac{1}{2}B(x, 2^j)$ , by using Lemma 3.3 and Lemma 2.8, we deduce that

$$\begin{aligned} I_2 &= \sum_{j=k_1+2}^{k_0-1} |[f_j(x) - f_{j+1}(x)] - [f_j(y) - f_{j+1}(y)]| \leq Cd(x, y) \left\{ \sum_{j=k_1+2}^{k_0-1} \frac{1}{2^j} \int_{B(x, 2^j)} |f_j - f_{j+1}| d\mu \right\} \\ &\leq Cd(x, y) \left\{ \sum_{j=k_1+2}^{k_0} \frac{1}{2^j} \int_{B(x, 2^j)} |f_j| d\mu \right\} \leq Cd(x, y) \left\{ \sum_{j=k_1+2}^{k_0} 2^j \left( \int_{B(x, 2^j)} |g|^{p_Q} d\mu \right)^{1/p_Q} \right\}. \end{aligned}$$

By Proposition 3.1 and Lemma 2.8, we see that for almost every  $z \in B(x, 2^{k_1+1})$ ,

$$\begin{aligned} |f_{k_1+2}(z)| &\leq C \sum_{k=-\infty}^{k_1+1} 2^{2k} \left( \int_{B(z, 2^k)} |g|^{p_Q} d\mu \right)^{1/p_Q} + C \int_{B(x, 2^{k_1+2})} |f_{k_1+2}| d\mu \\ &\leq C \sum_{k=-\infty}^{k_1+1} 2^{2k} \left( \int_{B(z, 2^k)} |g|^{p_Q} d\mu \right)^{1/p_Q} + C 2^{2k_1} \left( \int_{B(x, 2^{k_1+2})} |g|^{p_Q} d\mu \right)^{1/p_Q}, \end{aligned}$$

which together with the fact  $y \in B(x, 2^{k_1+1})$  implies that

$$I_3 \leq C 2^{k_1} \left\{ \sum_{k=-\infty}^{k_1+2} 2^k \left( \int_{B(x, 2^k)} |g|^{p_Q} d\mu \right)^{1/p_Q} + \sum_{k=-\infty}^{k_1+2} 2^k \left( \int_{B(y, 2^k)} |g|^{p_Q} d\mu \right)^{1/p_Q} \right\}.$$

Combining the estimates for the terms  $I_1$ ,  $I_2$  and  $I_3$ , and (3.2), we see that for almost all  $x, y \in B$  with  $d(x, y) < 1/2$ ,

$$|f(x) - f(y)| \leq Cd(x, y) \left\{ \frac{1}{r} \int_{2B} |f| d\mu + G_2(x) + G_2(y) \right\}.$$

For  $g \in L^\infty(2B)$ ,  $G_2 \in L^\infty(2B)$ , then up to a modification on a set with measure zero,  $f$  is a Lipschitz function on  $B$ .

By [76, Theorem 2.1], we see that for almost every  $x \in B$ ,

$$\begin{aligned} |\nabla f(x)| &\leq \text{Lip} f(x) = \limsup_{d(x, y) \rightarrow 0} \frac{|f(x) - f(y)|}{d(x, y)} \\ &\leq C \left\{ \frac{1}{r} \int_{B_{2R}(y_0)} |f| d\mu + \sum_{j=-\infty}^{k_0} 2^j \left( \int_{B(x, 2^j)} |g|^{p_Q} d\mu \right)^{1/p_Q} \right\}, \end{aligned}$$

proving the Theorem.  $\square$

We need the following potential estimate from Hajlasz and Koskela [57, Theorem 5.3]. Again,  $g$  refers both to a function defined on  $2B$  and to its zero extension to the exterior of  $2B$ .

**Theorem 3.4.** *Let  $(X, d, \mu, \mathcal{E})$  be a metric measure space satisfying  $(D_Q)$ ,  $Q > 1$ . Let  $B = B(x_0, r) \subset X$ ,  $g \in L^q(2B)$  with  $q > p_Q$ , and  $G$  be defined as (3.1). Then*

(i) *for  $q \in (p_Q, Q)$  and  $q^* = \frac{Qq}{Q-q}$ ,*

$$\|G_2\|_{L^{q^*}(B)} \leq Cr\mu(B)^{-1/Q}\|g\|_{L^q(2B)};$$

(ii) *for  $q > Q$*

$$\|G_2\|_{L^\infty(B)} \leq Cr\mu(B)^{-1/q}\|g\|_{L^q(2B)}.$$

Let us now turn to the case when only a reverse Hölder inequality  $(RH_p)$ ,  $p \in (2, \infty)$ , holds for gradients of harmonic functions. In this case, we do not know how to get pointwise estimates for the gradients of solutions to Poisson equations. As a substitute for this, we provide a quantitative  $L^p$ -estimate as follows.

**Theorem 3.5.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies  $(D_Q)$ ,  $Q > 1$ ,  $(P_{2, \text{loc}})$  and  $(UE)$ . Assume that  $(RH_p)$  hold for some  $p \in (2, \infty)$ . Then for every  $f \in W^{1,2}(2B)$ ,  $B = B(x_0, r)$ , satisfying  $\mathcal{L}f = g$  with  $g \in L^\infty(2B)$ , it holds for each  $q \in (p_Q, p]$  with  $\frac{1}{q} - \frac{1}{p} < \frac{1}{Q}$  that*

$$\left( \int_{B(x_0, r)} |\nabla f|^p d\mu \right)^{1/p} \leq \frac{C}{r} \left( \int_{B(x_0, 2r)} |f| d\mu + r^2 \left( \int_{B(x_0, 2r)} |g|^q d\mu \right)^{1/q} \right),$$

where  $C = C(p, q, C_Q, C_P(1), C_{LS})$  and  $p_Q$  is as in Definition 2.6.

We need the following well-known Christ's dyadic cube decomposition for metric measure spaces  $(X, d, \mu)$  from [32]; see also [64, Theorem 1.2].

**Lemma 3.6** (Christ's dyadic cubes). *There exists a collection of open subsets  $\{Q_\alpha^k \subset X : k \in \mathbb{Z}, \alpha \in I_k\}$ , where  $I_k$  denotes a certain (possibly finite) index set depending on  $k$ , and constants  $\delta \in (0, 1)$ ,  $a_0 \in (0, 1)$  and  $a_1 \in (0, \infty)$  such that*

- (i)  $\mu(X \setminus \cup_\alpha Q_\alpha^k) = 0$  for all  $k \in \mathbb{Z}$ ;
- (ii) if  $i > k$ , then either  $Q_\alpha^i \subset Q_\beta^k$  or  $Q_\alpha^i \cap Q_\beta^k = \emptyset$ ;
- (iii) for each  $k$  and all  $\alpha \neq \beta \in I_k$ ,  $Q_\alpha^k \cap Q_\beta^k = \emptyset$ ;
- (iv) for each  $(k, \alpha)$  and each  $i < k$ , there exists a unique  $\beta$  such that  $Q_\alpha^k \subset Q_\beta^i$ ;
- (v)  $\text{diam}(Q_\alpha^k) \leq a_1 \delta^k$ ;
- (vi) each  $Q_\alpha^k$  contains a ball  $B(z_\alpha^k, a_0 \delta^k)$ .

**Remark 3.7.** (i) In the above lemma, we can require  $\delta$  and  $a_1$  to be as small as we wish. This can be done by removing some generations, for instance,  $2k + 1$ -generations, from the set; see also [64, Theorem 1.2].

(ii) Under the doubling condition  $(D)$ , it is easy to see via conditions (iii) and (v) above that  $X = \cup_\alpha \bar{Q}_\alpha^k$  for each  $k$ .

The doubling condition allows us to conclude the following bounded overlap property.

**Proposition 3.8.** *Let  $(X, d, \mu)$  be a metric measure space satisfying  $(D_Q)$  for some  $Q > 1$ . For each  $\alpha$  and  $k$ , let  $B_\alpha^k = B(z_\alpha^k, a_1 \delta^k)$ . Then for each dilation  $t > 1$ , there exists a constant  $C(t, a_0, a_1, C_Q, Q) > 0$  such that for each  $k$ ,*

$$\sum_{\alpha} \chi_{tB_\alpha^k}(x) \leq C(t, a_1, C_Q, Q).$$

*Proof.* For each  $x \in X$ , let

$$C(x, k) = \sum_{\alpha} \chi_{tB_\alpha^k}(x).$$

Fix a point  $x_0$  and  $k \in \mathbb{Z}$ , and consider the ball  $B(x_0, 2ta_1 \delta^k)$ . Suppose that there exist  $C(x_0, k)$  balls  $\{B_{\alpha_j}^k\}_{j \leq C(x_0, k)}$  that are inside  $B(x_0, 2ta_1 \delta^k)$ . Using the doubling condition and the properties of the dyadic cubes from Lemma 3.6, we see that

$$\begin{aligned} V(x_0, 2ta_1 \delta^k) &\geq \sum_{j=1}^{C(x_0, k)} V(z_{\alpha_j}^k, a_0 \delta^k) \geq \sum_{j=1}^{C(x_0, k)} \frac{1}{C_Q(4ta_1)^Q} \mu(4tB_{\alpha_j}^k) \\ &\geq \frac{C(x_0, k)a_0^Q}{C_Q(4ta_1)^Q} V(x_0, 2ta_1 \delta^k). \end{aligned}$$

Therefore, we conclude that

$$C(x_0, k) \leq C_Q(4ta_1/a_0)^Q,$$

which completes the proof.  $\square$

We need the following geometric consequence of doubling; see [59] for instance.

**Lemma 3.9.** *Let  $(X, d, \mu)$  be a metric measure space satisfying  $(D)$ . Then there exists a constant  $N_\mu$  such that for each  $k \geq 1$ , every  $2^{-k}r$ -separated set in any ball  $B(x, r)$  in  $X$  has at most  $N_\mu^k$  elements.*

We shall make use of the Hardy-Littlewood maximal functions.

**Definition 3.10** (Hardy-Littlewood maximal function). *For any locally integrable function  $f$  on  $X$ , its Hardy-Littlewood maximal function is defined as*

$$\mathcal{M}f(x) := \sup_{B: x \in B} \int_B |f| d\mu,$$

where  $B$  is any ball. For  $p > 1$ , we define the  $p$ -Hardy-Littlewood maximal function as

$$\mathcal{M}_p f(x) := \sup_{B: x \in B} \left( \int_B |f|^p d\mu \right)^{1/p}.$$

Using the Poincaré inequality, it follows readily that  $\mathcal{M}_2(|\nabla f|)$  generates a Hajlasz gradient. The proof uses a telescopic argument, which is by now classical; see for instance [57] and the monographs [20, 59].

**Lemma 3.11.** *Assume that  $(X, d, \mu, \mathcal{E})$  satisfies (D) and  $(P_{2, \text{loc}})$ . Then for each  $\beta \in (0, 1)$  and  $r_0 > 0$ , there exists  $C = C(C_D, \beta, C_P(r_0)) > 0$  such that, for all  $f \in W^{1,2}(B)$ ,  $B = B(x_0, r)$ , it holds for almost all  $x, y \in \beta B$ ,*

$$|f(x) - f(y)| \leq Cd(x, y) (\mathcal{M}_2(|\nabla f| \chi_B)(x) + \mathcal{M}_2(|\nabla f| \chi_B)(y)).$$

Moreover, if  $f$  is continuous on  $B$ , then the above inequality holds for all  $x, y \in \beta B$ .

Let us now turn to the proof of the main gradient estimate. Recall that, under  $(D_Q)$  together with  $(P_{2, \text{loc}})$ , every solution  $f$  to the equation  $\mathcal{L}f = g$  with  $g \in L^\infty$  is locally Hölder continuous according to Lemma 2.9. Thus there exists a modification  $\tilde{f}$  such that  $\tilde{f} = f$  a.e., and every point is a Lebesgue point of  $\tilde{f}$ . Thus, in what follows, we may assume that every point is a Lebesgue point of our solution.

*Proof of Theorem 3.5.* For simplicity, we assume that  $a_1 = 1$  and  $\delta = \frac{1}{4}$  in Lemma 3.6.

We divide the proof into the following four steps.

**Step 1. Construction of a chain of balls.**

Let  $k_0 = \lceil -\log_4 r \rceil$  be the largest integer smaller than  $-\log_4 r$ , and set  $B^{k_0} = B(x_0, 6r)$ . For each  $k > k_0$ , let

$$I_{B,k} := \{\alpha : Q_\alpha^{k+2} \cap B(x_0, r) \neq \emptyset\},$$

and

$$\mathcal{F}_k = \{B_\alpha^k := B(z_\alpha^{k+2}, 2^{-2k}) : \alpha \in I_{B,k}\}.$$

Then, by Proposition 3.8, we see that for each  $k > k_0$ , it holds that

$$\sum_{\alpha \in I_{B,k} : B_\alpha^k \in \mathcal{F}_k} \chi_{B_\alpha^k}(x) \leq C_Q(64/a_0)^Q.$$

From the properties of dyadic cubes (Lemma 3.6), we see that:

- (i)  $B(x_0, r) \subset \cup_{\alpha \in I_{B,k}} B_\alpha^k$  for all  $k > k_0$ ;
- (ii) for each  $B_\alpha^k \in \mathcal{F}_k$ , there exist balls  $B_\alpha^j \in \mathcal{F}_j$ ,  $k_0 < j < k$ , such that for all  $k_0 < j < k$ ,  $B_\alpha^{j+1} \subset \frac{1}{3}B_\alpha^j$ , and  $B_\alpha^{k_0+1} \subset \frac{1}{3}B^{k_0} = 2B$ .

We call the collection  $B_\alpha^{k_0+1}, \dots, B_\alpha^{k-1}$  a chain associated to  $B_\alpha^k$  (and hence to  $Q_\alpha^k$ ).

**Proof of (ii):** If  $B_\alpha^k \in \mathcal{F}_k$ , then  $Q_\alpha^{k+2} \cap B(x_0, r) \neq \emptyset$ . Therefore, there exists  $Q_\alpha^{k+1}$  that contains  $Q_\alpha^{k+2}$  and hence,  $Q_\alpha^{k+1} \cap B(x_0, r) \neq \emptyset$  and  $d(z_\alpha^{k+2}, z_\alpha^{k+1}) \leq 2^{-2k-2}$  (by Lemma 3.6 (v)).

For each  $x \in B_\alpha^k$ ,

$$d(x, z_\alpha^{k+1}) < 2^{-2k-2} + 2^{-2k} = \frac{5}{4}2^{-2k} < \frac{5}{16}2^{-2k+2}.$$

From this, we conclude that  $\frac{1}{3}B_\alpha^{k-1} \supset B_\alpha^k$ .



In what follows, for each  $B_\alpha^k \in \mathcal{F}_k$ , we fix a chain from (ii).

**Step 2.** *Construction of a Hajlasz gradient via the chain.*

We first assume that  $\mathcal{L}f = g$  in  $6B = B^{k_0}$ ,  $B = B(x_0, r)$ , and  $g \in L^\infty(6B)$ . In the last step of the proof, we will complete the proof by using  $2B$  instead of  $6B$ .

Let  $f_{k_0} \in W_0^{1,2}(B(x_0, 6r))$  be the solution to

$$\mathcal{L}f_{k_0} = g$$

in  $B(x_0, 6r)$ , where the existence is guaranteed by Lemma 2.5. For each  $k > k_0$  and  $B_\alpha^k \in \mathcal{F}_k$ ,  $a \in I_{B,k}$ , we let  $f_{\alpha,k} \in W_0^{1,2}(B_\alpha^k)$  be the solution to the Poisson equation

$$\mathcal{L}f_{\alpha,k} = g$$

in  $B_\alpha^k$ . Then by Lemma 2.8, we see that for each  $k \geq k_0$  and  $a \in I_{B,k}$ ,

$$(3.3) \quad \int_{B_\alpha^k} |f_{\alpha,k}| d\mu \leq C 2^{-4k} \left( \int_{B_\alpha^k} |g|^{p_Q} \right)^{1/p_Q}.$$

In what follows, for consistency, we will write  $f_{k_0}$  as  $f_{\alpha,k_0}$ ,  $\alpha \in I_{B,k_0}$ , although there is only one element in  $I_{B,k_0}$ .

Define a function  $w_{k_0}$  on  $6B = B(x_0, 6r)$  by setting

$$w_{k_0}(x) = \mathcal{M}_2(|\nabla(f - f_{\alpha,k_0})| \chi_{3B})(x).$$

For each  $k > k_0$  and  $\alpha \in I_{B,k}$ , let  $\alpha' \in I_{B,k-1}$  the unique one such that  $Q_\alpha^{k+2} \subset Q_{\alpha'}^{k+1}$ . Define

$$w_k(x) := \sum_{\alpha \in I_{B,k}: B_\alpha^k \in \mathcal{F}_k} \mathcal{M}_2(|\nabla(f_{\alpha,k} - f_{\alpha',k-1})| \chi_{\frac{1}{2}B_\alpha^k})(x) \chi_{B_\alpha^k}(x)$$

on  $6B$ .

We also set

$$G(x) := \sum_{j=2k_0-4}^{\infty} 2^{-j} \left( \int_{B(x, 2^{-j})} |h|^{p_Q} d\mu \right)^{1/p_Q},$$

where  $p_Q$  is as in Definition 2.6 and  $h$  is the zero extension of  $g$  to  $X \setminus 6B$ .

*Claim:* There exists  $C = C(C_D, C_{LS}, C_P(1)) > 0$  such that for all  $x, y \in B(x_0, r)$  with  $d(x, y) < 1/2$ , it holds

$$(3.4) \quad |f(x) - f(y)| \leq C d(x, y) \left\{ G(x) + G(y) + \sum_{k \geq k_0} w_k(x) + \sum_{k \geq k_0} w_k(y) \right\}.$$

**Proof of the claim:** Let  $x, y \in B$  such that  $d(x, y) < 1/2$ . If  $d(x, y) \geq \frac{r}{64}$ , then

$$|f(x) - f(y)| \leq |(f - f_{\alpha,k_0})(x) - (f - f_{\alpha,k_0})(y)| + |f_{\alpha,k_0}(x) - f_{\alpha,k_0}(y)|,$$

where by Lemma 3.11 with  $(P_{2,\text{loc}})$  for balls with radii at most one and  $\beta = \frac{1}{2}$ , we have

$$\begin{aligned} & |(f - f_{\alpha,k_0})(x) - (f - f_{\alpha,k_0})(y)| \\ & \leq Cd(x, y) [\mathcal{M}_2(|\nabla(f - f_{\alpha,k_0})| \chi_{3B})(x) + \mathcal{M}_2(|\nabla(f - f_{\alpha,k_0})| \chi_{3B})(y)], \end{aligned}$$

and by Proposition 3.1 and (3.3) we have

$$|f_{\alpha,k_0}(x) - f_{\alpha,k_0}(y)| \leq Cd(x, y)[G(x) + G(y)].$$

The above two estimates complete the case  $d(x, y) \geq \frac{r}{64}$ .

Suppose now  $d(x, y) < \frac{r}{64}$  and  $d(x, y) < 1/2$ . Then there exists  $k > k_0$  such that  $1/2^{2k+6} \leq d(x, y) < 1/2^{2k+4}$ . From the properties of dyadic cubes, Lemma 3.6, we see that there exists a cube  $Q_\alpha^{k+2}$  such that  $x \in \bar{Q}_\alpha^{k+2}$ . Noticing that  $B_\alpha^k = B(z_\alpha^{k+2}, 2^{-2k})$ , we see that

$$d(y, z_\alpha^{k+2}) \leq d(x, y) + d(x, z_\alpha^{k+2}) < \frac{1}{2^{2k+4}} + \frac{1}{2^{2k+4}} = \frac{1}{2^{2k+3}},$$

and hence,  $x, y \in \frac{1}{3}B_\alpha^k$ .

Let  $\{B_\alpha^j \in \mathcal{F}_j\}_{k_0 \leq j < k}$  be the chain of balls such that  $\frac{1}{3}B_\alpha^j \supset B_\alpha^{j+1}$ , whose existence is guaranteed by **Step 1** (ii). Applying a telescopic argument, we obtain

$$\begin{aligned} |f(x) - f(y)| & \leq |(f - f_{\alpha,k_0})(x) - (f - f_{\alpha,k_0})(y)| + \sum_{j=k_0}^{k-1} |(f_{\alpha,j} - f_{\alpha,j+1})(x) - (f_{\alpha,j} - f_{\alpha,j+1})(y)| \\ & \quad + |f_{\alpha,k}(x) - f_{\alpha,k}(y)|. \end{aligned}$$

By using Lemma 3.11 with  $(P_{2,\text{loc}})$  for balls with radii at most one,  $\beta = \frac{2}{3}$  for  $k > k_0$  and  $\beta = \frac{1}{2}$  for  $k_0$ , we conclude that

$$\begin{aligned} & |(f - f_{\alpha,k_0})(x) - (f - f_{\alpha,k_0})(y)| + \sum_{j=k_0}^{k-1} |(f_{\alpha,j} - f_{\alpha,j+1})(x) - (f_{\alpha,j} - f_{\alpha,j+1})(y)| \\ & \leq Cd(x, y) \left\{ w_{k_0}(x) + w_{k_0}(y) + \sum_{k_0 < j \leq k} w_j(x) + \sum_{k_0 < j \leq k} w_j(y) \right\} \\ & \leq Cd(x, y) \left\{ \sum_{k \geq k_0} w_k(x) + \sum_{k \geq k_0} w_k(y) \right\}. \end{aligned}$$

On the other hand, by using (3.3), Proposition 3.1 and that  $d(x, y) \approx 2^{-2k}$ , we see that

$$\begin{aligned} & |f_{\alpha,k}(x) - f_{\alpha,k}(y)| \leq |f_{\alpha,k}(x)| + |f_{\alpha,k}(y)| \\ & \leq Cd(x, y) \left\{ \sum_{j=2k}^{\infty} 2^{-j} \left( \int_{B(x, 2^{-j})} |g|^{p_Q} d\mu \right)^{1/p_Q} + \sum_{j=2k}^{\infty} 2^{-j} \left( \int_{B(y, 2^{-j})} |g|^{p_Q} d\mu \right)^{1/p_Q} \right\} \end{aligned}$$

$$\leq Cd(x, y) \{G(x) + G(y)\}.$$

The above two estimates imply the claim.

**Step 3. Claim:** For  $q \in (p_Q, p]$  with  $1/q - 1/p < 1/Q$ , we have the estimate

$$\left\| G + \sum_{k \geq k_0} w_k \right\|_{L^p(B(x_0, r))} \leq \frac{C[V(x_0, 6r)]^{1/p}}{r} \left( \int_{B(x_0, 6r)} |f| d\mu + r^2 \left( \int_{B(x_0, 6r)} |g|^q d\mu \right)^{1/q} \right).$$

We begin by estimating the  $L^p$ -norm of the second term on the left-hand side. For a ball  $B_\alpha^k \in \mathcal{F}_k$ , let  $B_{\alpha'}^{k-1} = B_{\alpha'(\alpha)}^{k-1} \in \mathcal{F}_{k-1}$  be the ball from the definition of  $w_k$ ; then it satisfies  $\frac{1}{3}B_{\alpha'}^{k-1} \supset B_\alpha^k \in \mathcal{F}_k$ . Notice that  $f_{k,\alpha} - f_{\alpha,k-1}$  is harmonic on  $B_\alpha^k$ . Hence  $(RH_{p_0})$  (3.3) and the boundedness of the usual Hardy-Littlewood maximal operator on  $L^{p/2}$  with  $p > 2$  gives, for all  $k > k_0$  and  $\alpha \in I_{B,k}$ , that

$$\begin{aligned} \int_X \left[ \mathcal{M}_2(|\nabla(f_{\alpha,k} - f_{\alpha,k-1})| \chi_{\frac{1}{2}B_\alpha^k})(x) \chi_{B_\alpha^k}(x) \right]^p d\mu(x) &\leq \int_{\frac{1}{2}B_\alpha^k} |\nabla(f_{\alpha,k} - f_{\alpha,k-1})|^p d\mu(x) \\ &\leq C\mu(B_\alpha^k) \left( \frac{2^{2k}}{\mu(B_\alpha^k)} \int_{B_\alpha^k} |f_{\alpha,k} - f_{\alpha,k-1}| d\mu \right)^p \\ &\leq C\mu(B_{\alpha'}^{k-1}) 2^{-2pk} \left( \frac{1}{\mu(B_{\alpha'}^{k-1})} \int_{B_{\alpha'}^{k-1}} |g|^{p_Q} d\mu \right)^{p/p_Q}. \end{aligned}$$

Combining this with the fact that the sets  $\{B_\alpha^k\}_{\alpha \in I_{B,k}}$  have uniformly bounded overlaps for each  $k$ , we have that for each  $k > k_0$  and for each  $2 < p \leq p_0$ ,

$$\begin{aligned} \|w_k\|_{L^p(B(x_0, 6r))}^p &\leq C(\mu) \int_{B(x_0, 6r)} \left( \sum_{\alpha \in I_{B,k}: B_\alpha^k \in \mathcal{F}_k} \mathcal{M}_2(|\nabla(f_{\alpha,k} - f_{\alpha,k-1})| \chi_{\frac{1}{2}B_\alpha^k})(x) \chi_{B_\alpha^k}(x) \right)^p d\mu(x) \\ &\leq C(\mu) \sum_{\alpha \in I_{B,k}: B_\alpha^k \in \mathcal{F}_k} \int_{B_\alpha^k} \left( \mathcal{M}_2(|\nabla(f_{\alpha,k} - f_{\alpha,k-1})| \chi_{\frac{1}{2}B_\alpha^k})(x) \right)^p d\mu(x) \\ &\leq C(\mu) \sum_{\alpha \in I_{B,k}: B_\alpha^k \in \mathcal{F}_k} \mu(B_{\alpha'}^{k-1}) 2^{-2pk} \left( \frac{1}{\mu(B_{\alpha'}^{k-1})} \int_{B_{\alpha'}^{k-1}} |g|^{p_Q} d\mu \right)^{p/p_Q} \\ &\leq C(\mu) \sum_{\alpha \in I_{B,k}: B_\alpha^k \in \mathcal{F}_k} \mu(B_{\alpha'}^{k-1}) 2^{-2pk} \left( \frac{1}{\mu(B_{\alpha'}^{k-1})} \int_{B_{\alpha'}^{k-1}} |g|^q d\mu \right)^{p/q} \\ &\stackrel{\text{doubling}}{\leq} C(\mu) \sum_{\alpha \in I_{B,k}: B_\alpha^k \in \mathcal{F}_k} V(x_0, 6r)^{1-\frac{p}{q}} \frac{2^{2kQ(\frac{p}{q}-1)}}{r^{Q(\frac{p}{q}-1)}} 2^{-2pk} \left( \int_{B_{\alpha'}^{k-1}} |g|^q d\mu \right)^{p/q} \\ (3.5) \quad &\leq C(\mu) V(x_0, 6r)^{1-\frac{p}{q}} \frac{2^{2kQ(\frac{p}{q}-1)}}{r^{Q(\frac{p}{q}-1)}} 2^{-2pk} \left( \int_{B(x_0, 6r)} |g|^q d\mu \right)^{p/q}. \end{aligned}$$

Above the last inequality relies on the fact  $q \leq p$  and the fact that

$$\sum_{\alpha \in I_{B,k}: B_\alpha^k \in \mathcal{F}_k} \chi_{B_{\alpha'}^{k-1}}(x) \leq C(\mu, a_0).$$

Indeed, since  $B_\alpha^k \subset \frac{1}{3}B_{\alpha'}^{k-1}$ , we have  $Q_\alpha^{k+2} \subset \frac{1}{3}B_{\alpha'}^{k-1}$ . For each  $\alpha' \in I_{B,k-1}$ , let

$$I_{\alpha',k} := \left\{ \alpha : \alpha \in I_{B,k}, B_\alpha^k \subset \frac{1}{3}B_{\alpha'}^{k-1} \right\}.$$

By using dyadic cubes again, we see that

$$\mu(B_{\alpha'}^{k-1}) \geq \sum_{\alpha \in I_{\alpha',k}} V(z_\alpha^{k+2}, a_0 2^{-2k-4}) \geq \sum_{\alpha \in I_{\alpha',k}} \frac{C(\mu)}{(2^8 a_0)^Q} V(z_\alpha^{k+2}, 2^{-2k+4}) \geq \sum_{\alpha \in I_{\alpha',k}} \frac{C(\mu) a_0^Q}{(2^8)^Q} \mu(B_{\alpha'}^{k-1}),$$

which implies that  $\#(I_{\alpha',k}) \leq \frac{2^{8Q}}{C(\mu)(a_0)^Q}$ . Therefore, we conclude that

$$\sum_{\alpha \in I_{B,k}: B_\alpha^k \in \mathcal{F}_k} \chi_{B_{\alpha'}^{k-1}}(x) \leq \sum_{\alpha' \in I_{B,k-1}: B_{\alpha'}^{k-1} \in \mathcal{F}_{k-1}} \chi_{B_{\alpha'}^{k-1}}(x) \cdot \#(I_{\alpha',k}) \leq C(\mu, a_0).$$

By (3.5)

$$\|w_k\|_{L^p(B(x_0, 2r))} \leq C(\mu) V(x_0, 6r)^{\frac{1}{p} - \frac{1}{q}} \frac{2^{2kQ(\frac{1}{q} - \frac{1}{p})}}{r^{Q(\frac{1}{q} - \frac{1}{p})}} 2^{-2k} \left( \int_{B(x_0, 6r)} |g|^q d\mu \right)^{1/q}.$$

Therefore, by the Minkowski inequality,

$$\left\| \sum_{k > k_0} w_k \right\|_{L^p(B(x_0, r))} \leq C(\mu) V(x_0, 6r)^{\frac{1}{p} - \frac{1}{q}} r \left( \int_{B(x_0, 6r)} |g|^q d\mu \right)^{1/q}$$

provided  $p_Q < q \leq p$  and  $\frac{1}{q} - \frac{1}{p} < \frac{1}{Q}$ .

By applying Lemma 2.8 and  $(RH_p)$ , we conclude that

$$\begin{aligned} \|w_{k_0}\|_{L^p(B(x_0, r))} &\leq V(x_0, 2r)^{1/p} \frac{C}{r} \int_{B(x_0, 2r)} |f - f_{\alpha, k_0}| d\mu \\ (3.6) \quad &\leq \frac{CV(x_0, 2r)^{1/p}}{r} \left( \int_{B(x_0, 6r)} |f| d\mu + r^2 \left( \int_{B(x_0, 6r)} |g|^{p_Q} d\mu \right)^{1/p_Q} \right), \end{aligned}$$

which completes the estimate for  $L^p$ -integral of the second term on the left-hand side; recall that  $p_Q < q$ .

Regarding the first term, by Theorem 3.4 (cf. [57, Theorem 5.3]), we have

$$\|G\|_{L^p(B(x_0, r))} \leq C(\mu) r V(x_0, 6r)^{\frac{1}{p} - \frac{1}{q}} \left( \int_{B(x_0, 6r)} |g|^q d\mu \right)^{1/q}.$$

The claim then follows by combining the last inequality with (3.5) and (3.6).

**Step 4. Completion of the proof.**

For each  $y_0 \in B(x_0, r/2)$ , let  $0 \leq \psi_r \leq 1$  be a one-parameter family of Lipschitz cut-off functions such that

$$(3.7) \quad \begin{aligned} \psi_r(x) &= 1 \text{ whenever } x \in B(y_0, \min\{r/4, 1/8\}), \\ \psi_r(x) &= 0 \text{ whenever } x \in X \setminus B(y_0, \min\{r/2, 1/4\}), \text{ and } |\nabla \psi_r(x)| \leq \frac{C}{\min\{r, 1\}}. \end{aligned}$$

Then by **Step 2**, we see that, for almost all  $x, y \in B(y_0, \min\{r/2, 1/4\})$ ,

$$|(f\psi_r)(x) - (f\psi_r)(y)| \leq Cd(x, y) \left\{ \frac{|f(x)| + |f(y)|}{\min\{r, 1\}} + G(x) + G(y) + \sum_{k \geq k_0} w_k(x) + \sum_{k \geq k_0} w_k(y) \right\}.$$

Recall our assumption that  $g \in L^\infty(6B)$ . Therefore, by applying Lemma 2.5 to  $f_{\alpha, k_0}$ , Proposition 2.1 to  $f - f_{\alpha, k_0}$ , we see that  $f \in L^\infty(B)$ . This, together with **Step 3**, implies that  $f\psi_r \in W_0^{1,p}(B(y_0, \min\{r/2, 1/4\}))$ ; see [94, Theorem 4.8].

By (3.4) and the pointwise estimate of the gradient of Sobolev functions from Cheeger [25, Corollary 6.38] and Keith [73, Remark 2.16] (see also [76, Lemma 2.4]) for  $f\psi_r$ , we conclude that

$$|\nabla f(x)| = |\nabla(f\psi_r)(x)| \leq CG(x) + C \sum_{k \geq k_0} w_k(x)$$

for a.e.  $x \in B(y_0, \min\{r/4, 1/8\})$ . By the arbitrariness of  $y_0$ , we see this estimate holds for a.e.  $x \in B(x_0, r/2)$ . This together with the estimate from **Step 3** yields

$$(3.8) \quad |||\nabla f|||_{L^p(B(x_0, r/2))} \leq \frac{CV(x_0, 6r)^{1/p}}{r} \left( \int_{B(x_0, 6r)} |f| d\mu + r^2 \left( \int_{B(x_0, 6r)} |g|^q d\mu \right)^{1/q} \right).$$

Let us now replace  $B(x_0, 6r)$  on the R.H.S. by  $2B$ ,  $B = B(x_0, r)$ . By using Lemma 3.9, we see that  $B(x_0, r)$  contains at most  $N_\mu^5$  separate balls with radii  $r/32$ . Denote these balls by  $\{B(x_i, r/32)\}_{1 \leq i \leq N_\mu^5}$ . Then

$$B(x_0, r) \subset \bigcup_{1 \leq i \leq N_\mu^5} B(x_i, r/16).$$

By applying the estimate (3.8) to each  $B(x_i, \frac{12r}{16})$  yields

$$\begin{aligned} |||\nabla f|||_{L^p(B(x_i, r/16))} &\leq \frac{CV(x_i, \frac{12r}{16})^{1/p}}{r} \left( \int_{B(x_i, \frac{12r}{16})} |f| d\mu + r^2 \left( \int_{B(x_i, \frac{12r}{16})} |g|^q d\mu \right)^{1/q} \right) \\ &\leq \frac{CV(x_0, 2r)^{1/p}}{r} \left( \int_{B(x_0, 2r)} |f| d\mu + r^2 \left( \int_{B(x_0, 2r)} |g|^q d\mu \right)^{1/q} \right). \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} |||\nabla f|||_{L^p(B(x_0, r))} &\leq \sum_{i=1}^{N_\mu^5} |||\nabla f|||_{L^p(B(x_i, r/16))} \\ &\leq \frac{CV(x_0, 2r)^{1/p}}{r} \left( \int_{B(x_0, 2r)} |f| d\mu + r^2 \left( \int_{B(x_0, 2r)} |g|^q d\mu \right)^{1/q} \right), \end{aligned}$$

which completes the proof.  $\square$

### 3.2 Regularity of solutions to the Poisson equation

In this section, based on the a priori estimate from the preceding one, we give a quantitative estimate for solutions to the Poisson equation in full generality.

The first result concerns the case where harmonic functions are Lipschitz continuous.

**Theorem 3.12.** *Let  $(X, d, \mu, \mathcal{E})$  be a complete metric measure space, with  $\mu$  satisfying  $(D_Q)$ ,  $Q > 1$ ,  $(UE)$  and  $(P_{\infty, \text{loc}})$ . Assume that  $(RH_\infty)$  holds. Then for every  $f \in W^{1,2}(2B)$ ,  $B = B(x_0, r)$ , satisfying  $\mathcal{L}f = g$  with  $g \in L^q(2B)$ ,  $q > Q$ , we have*

$$|||\nabla f|||_{L^\infty(B)} \leq C \left\{ \frac{1}{r} \int_{2B} |f| d\mu + r \left( \int_{2B} |g|^q d\mu \right)^{1/q} \right\}.$$

where  $C = C(Q, C_Q, C_P) > 0$ .

*Proof.* If  $q = \infty$ , then the conclusion follows from Theorem 3.2.

Suppose now  $q \in (Q, \infty)$ . Let  $f_0 \in W_0^{1,2}(2B)$  be the solution to  $\mathcal{L}f_0 = g$  in  $2B$ . Moreover, for each  $k \in \mathbb{N}$ , let  $g_k := g\chi_{|g| \leq k}$ , and  $f_k \in W_0^{1,2}(2B)$  be the solution to  $\mathcal{L}f_k = g_k$  in  $2B$ . By using Lemma 2.5, Theorem 3.2 and Theorem 3.4, we conclude that

$$|||\nabla f_k|||_{L^\infty(B)} \leq C \left\{ \frac{1}{r} \int_{2B} |f_k| d\mu + r \|g_k\|_{L^q(2B)} \right\} \leq \frac{Cr}{\mu(B)^{1/q}} \|g_k\|_{L^q(2B)} \leq \frac{Cr}{\mu(B)^{1/q}} \|g\|_{L^q(2B)}.$$

On the other hand, notice that

$$\begin{aligned} \int_{2B} |\nabla(f_0 - f_k)|^2 d\mu &= \int_{2B} (f_0 g + f_k g_k - f_k g - f_0 g_k) d\mu \\ &\leq (\|f_0\|_{L^\infty(2B)} + \|f_k\|_{L^\infty(2B)}) \|g - g_k\|_{L^1(2B)} \rightarrow 0, \end{aligned}$$

as  $k \rightarrow \infty$ , which, together with the preceding inequality, implies that

$$|||\nabla f_0|||_{L^\infty(B)} \leq Cr \left( \int_{B(x_0, 2r)} |g|^q d\mu \right)^{1/q}.$$

Combining this with  $(RH_\infty)$  for  $f - f_0$  yields that

$$|||\nabla f|||_{L^\infty(B)} \leq C \left\{ \frac{1}{r} \int_{2B} |f| d\mu + r \left( \int_{B(x_0, 2r)} |g|^q d\mu \right)^{1/q} \right\},$$

as desired.  $\square$

We have the following quantitative  $L^p$ -estimate for solutions to the Poisson equation.

**Theorem 3.13.** *Let  $(X, d, \mu, \mathcal{E})$  be a complete metric measure space, with  $\mu$  satisfying  $(D_Q)$ ,  $Q > 1$ ,  $(UE)$  and  $(P_{2,\text{loc}})$ . Let  $p \in (2, \infty)$ . Assume  $(RH_p)$  holds. Let  $p_Q$  be as in Definition 2.6 and  $q \in (p_Q, p]$  with  $\frac{1}{q} - \frac{1}{p} < \frac{1}{Q}$ . Then for every  $f \in W^{1,2}(2B)$ ,  $B = B(x_0, r)$ , satisfying  $\mathcal{L}f = g$  with  $g \in L^q(2B)$ ,*

$$\left( \int_{B(x_0, r)} |\nabla f|^p d\mu \right)^{1/p} \leq \frac{C}{r} \left( \int_{B(x_0, 2r)} |f| d\mu + r^2 \left( \int_{B(x_0, 2r)} |g|^q d\mu \right)^{1/q} \right),$$

where  $C = C(p, q, C_D, C_P(1), C_{LS})$ .

*Proof.* For each  $k \in \mathbb{N}$ , let  $g_k := g\chi_{|g| \leq k}$ , and let  $f_k \in W_0^{1,2}(2B)$  be the solution to  $\mathcal{L}f_k = g_k$  in  $2B$ .

By Lemma 2.8, we see that there exists a solution  $f_0 \in W_0^{1,2}(2B)$  to  $\mathcal{L}f_0 = g$  in  $2B$ , with

$$(3.9) \quad \int_{2B} |f_0| d\mu \leq Cr \left( \int_{2B} |\nabla f_0|^2 d\mu \right)^{1/2} \leq Cr^2 \left( \int_{2B} |g|^q d\mu \right)^{1/q}.$$

For each  $k \in \mathbb{N}$ , by using Lemma 2.8 again, we obtain

$$\int_{2B} |f_0 - f_k| d\mu \leq Cr \left( \int_{2B} |\nabla(f_0 - f_k)|^2 d\mu \right)^{1/2} \leq Cr^2 \left( \int_{2B} |g - g_k|^q d\mu \right)^{1/q},$$

since  $f_0 - f_k \in W_0^{1,2}(2B)$ . From this, we conclude that  $f_k \rightarrow f_0$  in  $W_0^{1,2}(2B)$ .

By Lemma 2.8 and Theorem 3.5, we have for each  $k \in \mathbb{N}$  that

$$\begin{aligned} \left( \int_{B(x_0, r)} |\nabla f_k|^p d\mu \right)^{1/p} &\leq \frac{C}{r} \left( \int_{B(x_0, 2r)} |f_k| d\mu + r^2 \left( \int_{B(x_0, 2r)} |g_k|^q d\mu \right)^{1/q} \right) \\ &\leq Cr \left( \int_{B(x_0, 2r)} |g_k|^q d\mu \right)^{1/q} \leq Cr \left( \int_{B(x_0, 2r)} |g|^q d\mu \right)^{1/q}. \end{aligned}$$

Letting  $k \rightarrow \infty$ , we conclude that

$$\left( \int_{B(x_0, r)} |\nabla f_0|^p d\mu \right)^{1/p} \leq Cr \left( \int_{B(x_0, 2r)} |g|^q d\mu \right)^{1/q}.$$

By applying this together with  $(RH_p)$  for the harmonic function  $f - f_0$  on  $B(x_0, r)$  and (3.9) we obtain

$$\begin{aligned} \left( \int_{B(x_0, r)} |\nabla f|^p d\mu \right)^{1/p} &\leq \left( \int_{B(x_0, r)} |\nabla(f - f_0)|^p d\mu \right)^{1/p} + \left( \int_{B(x_0, r)} |\nabla f_0|^p d\mu \right)^{1/p} \\ &\leq \frac{C}{r} \int_{B(x_0, 2r)} |f - f_0| d\mu + Cr \left( \int_{B(x_0, 2r)} |g|^q d\mu \right)^{1/q} \\ &\leq \frac{C}{r} \int_{B(x_0, 2r)} |f| d\mu + Cr \left( \int_{B(x_0, 2r)} |g|^q d\mu \right)^{1/q}, \end{aligned}$$

as desired.  $\square$

The above two theorems yield the following quantitative Hölder regularity of solutions to the Poisson equation.

**Corollary 3.14.** *Let  $(X, d, \mu, \mathcal{E})$  be a complete metric measure space satisfying  $(D_Q)$ ,  $Q > 1$ , and  $(UE)$ . Assume that  $(RH_p)$  and  $(P_{2, \text{loc}})$  hold for some  $p \in (Q, \infty]$ . Let  $q > \max\{Q/2, 1\}$  and  $\alpha := \alpha(p, q) = \min\{1 - Q/p, 2 - Q/q\}$ . If  $\mathcal{L}f = g$  in  $B(x_0, r)$  with  $g \in L^q(B)$ , then  $f$  belongs to  $C_{\text{loc}}^\alpha(B)$ .*

## 4 Elliptic equations vs parabolic equations

### 4.1 From elliptic equations to parabolic equations

In this section, we give quantitative gradient estimates for the heat kernel by using the regularity of solutions to the Poisson equation.

To begin with, let us recall that, under  $(D)$  and  $(UE)$ , we have estimate for the time derivative of heat kernel as

$$(4.1) \quad \left| \frac{\partial h_t}{\partial t}(x, y) \right| \leq \frac{C}{t V(y, \sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{ct} \right\},$$

for all  $t > 0$ ; see [21, 92, 100, 101].

A version of the following result, requiring a slightly stronger condition  $(P_2)$ , has been established in [69]. The following proof is the same as that of [69, Theorem 3.2], but since the assumptions here is different from [69, Theorem 3.2], we give a fully adapted proof for the sake of completeness.

**Proposition 4.1.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies  $(D)$ ,  $(UE)$  and  $(P_{\infty, \text{loc}})$ . Then  $(RH_\infty)$  implies  $(GLY_\infty)$ .*

*Proof.* Notice that as the heat kernel is the fundamental solution for the heat equation, it always holds that

$$-\mathcal{L}h_t = \frac{\partial}{\partial t}h_t, \quad \mu \text{ a.e.}$$

Fix a  $t > 0$ . Consider the equation  $-\mathcal{L}h_t(\cdot, y) = \frac{\partial}{\partial t}h_t(\cdot, y)$  on the ball  $B(x, \sqrt{t})$ . Notice that for each fixed  $t > 0$ ,  $\frac{\partial}{\partial t}h_t$  is a bounded function on the ball  $B(x, 2\sqrt{t})$ . Thus, by Theorem 3.2, we see that for almost all  $x, y$ ,

$$\begin{aligned} |\nabla_x h_t(x, y)| &\leq C \left\{ \frac{1}{\sqrt{t}} \int_{B(x, 2\sqrt{t})} h_t(z, y) d\mu(z) \right. \\ &\quad \left. + \sum_{j=-\infty}^{\lfloor \log_2 2\sqrt{t} \rfloor} 2^j \left( \int_{B(x, 2^j)} \left| \frac{\partial}{\partial t} h_t(z, y) \right|^{p_Q} d\mu(z) \right)^{1/p_Q} \right\} \end{aligned}$$



We divide the proof into two cases, i.e.,  $d^2(x, y) > 16t$  and  $d^2(x, y) \leq 16t$ , and first consider  $x, y \in X$  with  $d^2(x, y) > 16t$ . Notice that for every  $z \in B(x, 2\sqrt{t})$ ,

$$d(y, z) \geq d(x, y) - d(x, z) \geq d(x, y) - d(x, y)/2 = d(x, y)/2.$$

From this, together with (UE), we see that

$$\frac{1}{\sqrt{t}} \int_{B(x, 2\sqrt{t})} h_t(z, y) d\mu(z) \leq \frac{C}{\sqrt{t}V(x, 2\sqrt{t})} \int_{B(x, 2\sqrt{t})} \frac{e^{-\frac{d^2(z, y)}{ct}}}{V(y, \sqrt{t})} d\mu(z) \leq \frac{C}{\sqrt{t}V(y, \sqrt{t})} e^{-\frac{d^2(x, y)}{ct}}.$$

By using (4.1) and the doubling condition, we also obtain

$$\begin{aligned} & \sum_{j=-\infty}^{\lceil \log_2 2\sqrt{t} \rceil} 2^j \left( \int_{B(x, 2^j)} \left| \frac{\partial}{\partial t} h_t(z, y) \right|^{p_Q} d\mu(z) \right)^{1/p_Q} \\ & \leq \sum_{j=-\infty}^{\lceil \log_2 2\sqrt{t} \rceil} 2^j \left( \int_{B(x, 2^j)} \left| \frac{C}{t} \frac{1}{V(y, \sqrt{t})} \exp \left\{ -\frac{d^2(z, y)}{ct} \right\} \right|^{p_Q} d\mu(z) \right)^{1/p_Q} \\ & \leq \frac{C}{t} \frac{1}{V(y, \sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{ct} \right\} \sum_{j=-\infty}^{\lceil \log_2 2\sqrt{t} \rceil} 2^j \\ & \leq \frac{C}{\sqrt{t}} \frac{1}{V(y, \sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{ct} \right\}. \end{aligned}$$

Combining the above two estimates, we conclude that

$$(4.2) \quad |\nabla_x h_t(x, y)| \leq \frac{C}{\sqrt{t}} \frac{1}{V(y, \sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{ct} \right\}.$$

When  $x, y \in X$  with  $d^2(x, y) \leq 16t$ , the exponential term  $\exp\{-\frac{d^2(x, y)}{ct}\}$  is equivalent to 1. Applying the proof for (4.2) and discarding the exponential term, we arrive at

$$|\nabla_x h_t(x, y)| \leq \frac{C}{\sqrt{t}} \frac{1}{V(y, \sqrt{t})},$$

which together with (4.2) implies that for almost all  $x, y \in X$ ,

$$(4.3) \quad |\nabla_x h_t(x, y)| \leq \frac{C}{\sqrt{t}} \frac{1}{V(y, \sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{ct} \right\}.$$

The proof is complete.  $\square$

The following result was well known; see for instance [7].

**Proposition 4.2.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies (D) and (UE). Then  $(GLY_\infty)$  implies  $(G_p)$  for all  $p \in [1, \infty]$ .*

*Proof.* By decomposing  $X$  into the union of  $B(x, \sqrt{t})$  and  $B(x, 2^k \sqrt{t}) \setminus B(x, 2^{k-1} \sqrt{t})$  for each  $k \geq 1$ , one sees via (D) that

$$(4.4) \quad \int_X \frac{1}{V(x, \sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{ct} \right\} d\mu(y) \leq C(C_D).$$

The conclusion then follows from this and  $(GLY_\infty)$ ; see [7] for instance.  $\square$

We will also need the following observation.

**Proposition 4.3.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies (D) and (UE). Suppose that  $(P_{2, \text{loc}})$  and  $(RH_p)$  hold for some  $p \in (2, \infty)$ . Then  $(GLY_p)$  holds.*

*Proof.* Decompose the space  $X$  into  $B = B(y, 2\sqrt{t})$  and  $B(y, 2^{k+1}\sqrt{t}) \setminus B(y, 2^k\sqrt{t})$ ,  $k \geq 1$ . Denote  $B(y, 2^{k+1}\sqrt{t}) \setminus B(y, 2^k\sqrt{t})$  by  $U_k(B)$ . By Theorem 3.5, (UE) and (4.1), we see that

$$\begin{aligned} & \| |\nabla_x h_t(\cdot, y)| \|_{L^p(B)} \\ & \leq \frac{CV(y, 4\sqrt{t})^{1/p}}{\sqrt{t}} \left( \int_{B(y, 4\sqrt{t})} |h_t(x, y)| d\mu(x) + t \left( \int_{B(y, 4\sqrt{t})} \left| \frac{\partial}{\partial t} h_t(x, y) \right|^p d\mu(x) \right)^{1/p} \right) \\ & \leq \frac{C}{\sqrt{t}} V(y, \sqrt{t})^{1/p-1}. \end{aligned}$$

Let  $\{B_{k,j} = B(x_{k,j}, \sqrt{t}/2)\}_j$  be a maximal set of pairwise disjoint balls with radius  $2^{-1}\sqrt{t}$  in  $B(y, 2^{k+1}\sqrt{t})$ . Then it is easy to see that

$$B(y, 2^{k+1}\sqrt{t}) \subset \cup_j B(x_{k,j}, \sqrt{t})$$

and

$$\sum_j \chi_{4B_{k,j}}(x) \leq C(C_D).$$

Therefore, by applying Theorem 3.13, (D), (UE), and (4.1), we conclude that

$$\begin{aligned} & \int_{U_k(B)} |\nabla_x h_t(x, y)|^p d\mu(x) \\ & \leq \sum_{j: 2B_{k,j} \cap U_k(B) \neq \emptyset} \int_{2B_{k,j}} |\nabla_x h_t(x, y)|^p d\mu(x) \\ & \leq \sum_{j: 2B_{k,j} \cap U_k(B) \neq \emptyset} \frac{C\mu(4B_{k,j})}{t^{p/2}} \left( \int_{4B_{k,j}} |h_t(x, y)| d\mu(x) + t \left( \int_{4B_{k,j}} \left| \frac{\partial}{\partial t} h_t(x, y) \right|^p d\mu(x) \right)^{1/p} \right)^p \\ & \leq \sum_{j: 2B_{k,j} \cap U_k(B) \neq \emptyset} \frac{C\mu(4B_{k,j})}{t^{p/2}} V(y, \sqrt{t})^{-p} \exp \left\{ \frac{-c2^{2k}t}{t} \right\} \\ & \leq CV(y, 2^{k+2}\sqrt{t}) \frac{\exp \{-c2^{2k}\}}{t^{p/2} V(y, \sqrt{t})^p} \leq CV(y, \sqrt{t}) 2^{kQ} \frac{\exp \{-c2^{2k}\}}{t^{p/2} V(y, \sqrt{t})^p} \end{aligned}$$

$$\leq C \frac{\exp\{-c2^{2k}\}}{t^{p/2}V(y, \sqrt{t})^{p-1}}.$$

This together with the estimate on  $\|\nabla_x h_t(\cdot, y)\|_{L^p(B)}$  from the beginning of the proof allow us to deduce that there exists  $\gamma > 0$  such that

$$\int_X |\nabla_x h_t(x, y)|^p \exp\{\gamma d^2(x, y)/t\} d\mu(x) \leq \frac{C}{t^{p/2}V(y, \sqrt{t})^{p-1}},$$

which completes the proof.  $\square$

The following statement is contained in [7, p. 944]. We repeat the proof for the sake of completeness.

**Proposition 4.4.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies (D) and (UE). Then  $(GLY_p)$  for some  $p \in (2, \infty)$  implies  $(G_p)$ .*

*Proof.* Applying  $(GLY_p)$ , the Hölder inequality, (UE) and (4.4), we deduce that for each  $f \in L^p(X, \mu)$ ,

$$\begin{aligned} \|\nabla H_t f\|_p^p &\leq \int_X \left( \int_X |\nabla_x h_t(x, y)| |f(y)| d\mu(y) \right)^p d\mu(x) \\ &\leq \int_X \left( \int_X |\nabla_x h_t(x, y)|^p \exp\{\gamma d^2(x, y)/t\} V(y, \sqrt{t})^{p-1} |f(y)|^p d\mu(y) \right) \\ &\quad \times \left( \int_X \frac{1}{V(y, \sqrt{t})} \exp\left\{-\frac{p'\gamma d^2(x, y)}{pt}\right\} d\mu(y) \right)^{1/p'} d\mu(x) \\ &\stackrel{(4.4)}{\leq} \int_X \int_X |\nabla_x h_t(x, y)|^p \exp\{\gamma d^2(x, y)/t\} V(y, \sqrt{t})^{p-1} |f(y)|^p d\mu(y) d\mu(x) \\ &\leq \frac{C}{t^{p/2}} \|f\|_p^p. \end{aligned}$$

This yields  $(G_p)$ .  $\square$

**Remark 4.5.** If  $(P_p)$  holds, then one can also use the open-ended property of the reverse Hölder inequality  $(RH_p)$  (Lemma 6.2 below), Theorem 3.5 and the Hardy-Littlewood maximal operator to prove the fact that  $(RH_p)$  ( $p \in (2, \infty)$ ) yields  $(G_p)$ . We will not go through this argument and leave the details to interested readers.

## 4.2 From parabolic equations to elliptic equations

In this section, we show that  $(G_p)$  implies  $(RH_p)$ . We begin with an abstract reproducing formula for harmonic functions.

**Lemma 4.6** (Reproducing formula). *Let  $(X, d, \mu, \mathcal{E})$  be a non-compact metric measure Dirichlet space endowed with a “carré du champ”. Assume that (D) and (UE) hold. Let  $\Phi \in \mathcal{S}(\mathbb{R})$  be an even function whose Fourier transform  $\hat{\Phi}$  satisfies  $\text{supp } \hat{\Phi} \subset [-1, 1]$  and  $\Phi(0) = 1$ . Then if  $u \in W^{1,2}(3B)$  is harmonic on  $3B$ ,  $B = B(x_0, r)$ , for each  $0 < t \leq 1$ ,  $u = \Phi(tr\sqrt{\mathcal{L}})u$  as functions in  $W^{1,2}(B)$ .*

*Proof.* Since  $\Phi'(0) = 0$ , note that the function  $\tilde{\Phi}(s) := s^{-1}\Phi'(s) \in \mathcal{S}(\mathbb{R})$  extends to an analytic function which satisfies a Paley-Wiener estimate with the same exponent as  $\Phi$ . By applying Lemma 2.15 to the functions  $t^{2\kappa}\Phi(t)$ ,  $\kappa \in \mathbb{Z}_+$ , and  $\tilde{\Phi}$ , we conclude that the operators  $(t^2\mathcal{L})^\kappa\Phi(t\sqrt{\mathcal{L}})$  and  $(t^2\mathcal{L})^{-1/2}\Phi'(t\sqrt{\mathcal{L}})$  satisfy

$$(4.5) \quad \int_X \langle (t^2\mathcal{L})^\kappa\Phi(t\sqrt{\mathcal{L}})f_1, f_2 \rangle d\mu = 0,$$

and

$$(4.6) \quad \int_X \langle (t^2\mathcal{L})^{-1/2}\Phi'(t\sqrt{\mathcal{L}})f_1, f_2 \rangle d\mu = 0,$$

for all  $0 < t < d(E, F)$  with  $E, F \subset X$ ,  $f_1 \in L^2(E)$ , and  $f_2 \in L^2(F)$ .

Let  $\psi$  be a Lipschitz cut-off function such that  $\psi = 1$  on  $\frac{8}{3}B$ ,  $\psi = 0$  outside  $3B$ . Let  $\varepsilon \in (0, r/4)$ . Then for each  $g \in L^2(\frac{3}{2}B)$  with support in  $\overline{\frac{3}{2}B}$ , we have

$$\Phi(\varepsilon\sqrt{\mathcal{L}})g \in \mathcal{D}(\mathcal{L})$$

with support in  $\overline{\frac{7}{4}B}$ . Since  $\Phi(0) = 1$ , we have

$$1 - \Phi(r\sqrt{\mathcal{L}}) = - \int_0^r \sqrt{\mathcal{L}}\Phi'(s\sqrt{\mathcal{L}}) ds,$$

which together with (4.6) implies that

$$(4.7) \quad \int_X \langle (t^2r^2\mathcal{L})^{-1}(1 - \Phi(tr\sqrt{\mathcal{L}}))f_1, f_2 \rangle d\mu = 0,$$

for all  $0 < t < d(E, F)$  with  $E, F \subset X$ ,  $f_1 \in L^2(E)$ , and  $f_2 \in L^2(F)$ . Therefore, for each  $t \leq 1$ , we have

$$(4.8) \quad (t^2r^2\mathcal{L})^{-1}(1 - \Phi(tr\sqrt{\mathcal{L}}))\Phi(\varepsilon\sqrt{\mathcal{L}})g \in \mathcal{D}(\mathcal{L})$$

with support in  $\overline{\frac{11}{4}B}$ . From this together with  $\mathcal{L}u = 0$  in  $3B$ , we conclude that

$$\begin{aligned} \int_X \langle (1 - \Phi(tr\sqrt{\mathcal{L}}))u, \Phi(\varepsilon\sqrt{\mathcal{L}})g \rangle d\mu &= \int_X \langle u, (1 - \Phi(tr\sqrt{\mathcal{L}}))\Phi(\varepsilon\sqrt{\mathcal{L}})g \rangle d\mu \\ &= \int_X \langle u\psi, (1 - \Phi(tr\sqrt{\mathcal{L}}))\Phi(\varepsilon\sqrt{\mathcal{L}})g \rangle d\mu \end{aligned}$$

$$\begin{aligned}
&= r^2 \int_{3B} \langle \nabla u, \nabla(r^2 \mathcal{L})^{-1}(1 - \Phi(tr \sqrt{\mathcal{L}}))\Phi(\varepsilon \sqrt{\mathcal{L}})g \rangle d\mu \\
&= 0.
\end{aligned}$$

Since  $g$  is arbitrary, and by Lemma 2.16  $\Phi(\varepsilon \sqrt{\mathcal{L}})g \rightarrow g$  in  $L^2(X, \mu)$  as  $\varepsilon \rightarrow 0$ , we find that  $(1 - \Phi(tr \sqrt{\mathcal{L}}))u = 0$  in  $L^2(B)$ . Hence  $u(x) = \Phi(tr \sqrt{\mathcal{L}})u(x)$  for a.e.  $x \in \frac{3}{2}B$ .

On the other hand, notice that for each  $\varphi \in W_0^{1,2}(\frac{3}{2}B)$ , it holds that

$$\begin{aligned}
(4.9) \quad \int_X \langle \mathcal{L}\Phi(tr \sqrt{\mathcal{L}})u, \varphi \rangle d\mu &= \int_X \langle u, \mathcal{L}\Phi(tr \sqrt{\mathcal{L}})\varphi \rangle d\mu = \int_X \langle u\psi, \mathcal{L}\Phi(tr \sqrt{\mathcal{L}})\varphi \rangle d\mu \\
&= \int_{3B} \langle \nabla u, \nabla\Phi(tr \sqrt{\mathcal{L}})\varphi \rangle d\mu = 0,
\end{aligned}$$

since  $\Phi(tr \sqrt{\mathcal{L}})\varphi$  is supported in  $\overline{\frac{3}{2}B}$ . Therefore, we can conclude that  $\Phi(tr \sqrt{\mathcal{L}})u$  is harmonic on  $\frac{3}{2}B$ .

Using the fact  $u(x) = \Phi(tr \sqrt{\mathcal{L}})u(x)$  for a.e.  $x \in \frac{3}{2}B$  for  $t \leq 1$ , and applying the Caccioppoli inequality Lemma 2.4 to  $u - \Phi(tr \sqrt{\mathcal{L}})u$  on  $\frac{3}{2}B$ , one can conclude that

$$\int_B |\nabla(u - \Phi(tr \sqrt{\mathcal{L}})u)|^2 d\mu \leq \frac{C}{r^2} \int_{\frac{3}{2}B} |u - \Phi(tr \sqrt{\mathcal{L}})u|^2 d\mu = 0,$$

and hence,  $u = \Phi(tr \sqrt{\mathcal{L}})u$  in  $W^{1,2}(B)$  for each  $t \leq 1$ . The proof is complete.  $\square$

**Remark 4.7.** Notice that, for each  $f \in L^2(X, \mu)$ ,  $\Phi(r \sqrt{\mathcal{L}})f \in W^{1,2}(X)$  and

$$\|\nabla\Phi(r \sqrt{\mathcal{L}})f\|_2 = \|\sqrt{\mathcal{L}}\Phi(r \sqrt{\mathcal{L}})f\|_2 \leq \frac{C}{r}\|f\|_2,$$

see Lemma 2.14.

**Corollary 4.8.** Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies (D) and (UE). Let  $\Phi \in \mathcal{S}(\mathbb{R})$  be an even function whose Fourier transform  $\hat{\Phi}$  satisfies  $\text{supp } \hat{\Phi} \subset [-1, 1]$  and  $\Phi(0) = 1$ . Then if  $u \in W^{1,2}(X)$  is harmonic on  $3B$ ,  $B = B(x_0, r)$ , for each  $0 < t \leq 1$ ,  $u$  equals  $\Phi(tr \sqrt{\mathcal{L}})u$  as functions in  $W^{1,2}(B)$ .

*Proof.* Notice that by Lemma 4.6, for each  $0 < t \leq 1$ ,  $u(x) = \Phi(tr \sqrt{\mathcal{L}})(u\chi_{3B})(x)$ , a.e.  $x \in B$ . On the other hand, by (4.5), we see that

$$\Phi(tr \sqrt{\mathcal{L}})(u\chi_{X \setminus 3B})(x) = 0$$

on  $B$ , which allows us to conclude that for each  $0 < t \leq 1$ ,  $u = \Phi(tr \sqrt{\mathcal{L}})u$  in  $W^{1,2}(B)$ .  $\square$

The main result of this section reads as follows.

**Theorem 4.9.** Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies (D) and (UE). If  $(G_{p_0})$  holds for some  $p_0 \in (2, \infty]$ , then  $(RH_{p_0})$  holds.

*Proof.* Let  $\Phi \in \mathcal{S}(\mathbb{R})$  be an even function whose Fourier transform  $\hat{\Phi}$  satisfies  $\text{supp } \hat{\Phi} \subset [-1/2, 1/2]$  and  $\Phi(0) = 1$ . Then it follows that  $\Phi^2 \in \mathcal{S}(\mathbb{R})$  and  $\text{supp } \hat{\Phi}^2 \subset [-1, 1]$ . In the proof, for simplicity we denote  $V(x, r)$  by  $V_r(x)$ .

**Step 1.** *Boundedness of the spectral multipliers.*

**Claim 1.** We first claim that, for each  $p \in [1, 2]$ , there exists  $C > 0$  such that

$$\sup_{r>0} \|V_r^{1/p-1/2} \Phi(r\sqrt{\mathcal{L}})\|_{p \rightarrow 2} \leq C.$$

By [21, Proposition 4.1.1] and the fact  $\sup_{t>0} |\Phi(t)(1+t^2)^N| < \infty$ , one has

$$\begin{aligned} \|V_r^{1/p-1/2} \Phi(r\sqrt{\mathcal{L}})\|_{p \rightarrow 2} &= \|V_r^{1/p-1/2} \Phi(r\sqrt{\mathcal{L}})(1+r^2\mathcal{L})^N V_r^{1/2-1/p} V_r^{1/p-1/2} (1+r^2\mathcal{L})^{-N}\|_{p \rightarrow 2} \\ &\leq C \|\Phi(r\sqrt{\mathcal{L}})(1+r^2\mathcal{L})^N\|_{2 \rightarrow 2} \|V_r^{1/p-1/2} (1+r^2\mathcal{L})^{-N}\|_{p \rightarrow 2} \\ &\leq C \|V_r^{1/p-1/2} (1+r^2\mathcal{L})^{-N}\|_{p \rightarrow 2}, \end{aligned}$$

where we choose  $N > Q$ ,  $Q$  is the number from  $(D_Q)$ . Notice that for any  $f \in L^p(X, \mu)$  one has

$$\begin{aligned} \|V_r^{1/p-1/2} (1+r^2\mathcal{L})^{-N} f\|_2 &\leq \frac{C}{\Gamma(N)} \int_0^\infty \left( \int_X \left| e^{-s} s^{N-1} V_r(x)^{1/p-1/2} e^{-sr^2\mathcal{L}} f(x) \right|^2 d\mu(x) \right)^{1/2} ds \\ &\leq C \int_0^\infty e^{-s} s^{N-1} \|V_r^{1/p} e^{-sr^2\mathcal{L}} f\|_\infty^{1-p/2} \left( \int_X \left| e^{-sr^2\mathcal{L}} f(x) \right|^p d\mu(x) \right)^{1/2} ds \\ &\leq C \int_0^\infty e^{-s} s^{N-1} \left\| \frac{V_r}{V_{\sqrt{sr}}} \right\|_\infty^{1/p-1/2} \|f\|_p^{1-p/2} \|f\|_p^{p/2} ds \\ &\leq C \int_0^\infty e^{-s} s^{N-1} \frac{1}{(s \wedge 1)^{Q/2(1/p-1/2)}} \|f\|_p ds \\ &\leq C \|f\|_p, \end{aligned}$$

which proves the claim.

**Claim 2.** For each  $p \in (2, \infty]$ , if  $(G_p)$  holds, then there exists  $C > 0$  such that

$$\sup_{r>0} \|rV_r^{1-1/p} |\nabla \Phi(r\sqrt{\mathcal{L}})|^2\|_{1 \rightarrow p} \leq C.$$

By Claim 1 and [21, Proposition 4.1.1] again, we have

$$\begin{aligned} \|rV_r^{1-1/p} |\nabla \Phi(r\sqrt{\mathcal{L}})|^2\|_{1 \rightarrow p} &= \|rV_r^{1-1/p} |\nabla \Phi(r\sqrt{\mathcal{L}})| V_r^{-1/2} V_r^{1/2} \Phi(r\sqrt{\mathcal{L}})\|_{1 \rightarrow p} \\ &\leq C \|rV_r^{1-1/p} |\nabla \Phi(r\sqrt{\mathcal{L}})| V_r^{-1/2}\|_{2 \rightarrow p} \|V_r^{1/2} \Phi(r\sqrt{\mathcal{L}})\|_{1 \rightarrow 2} \\ &\leq C \|rV_r^{1-1/p} |\nabla \Phi(r\sqrt{\mathcal{L}})| V_r^{-1/2}\|_{2 \rightarrow p} \\ &\leq Cr \| |\nabla \Phi(r\sqrt{\mathcal{L}})| V_r^{1/2-1/p} \|_{2 \rightarrow p} \\ &\leq Cr \| |\nabla (1+r^2\mathcal{L})^{-1}| \|_{p \rightarrow p} \| (1+r^2\mathcal{L}) \Phi(r\sqrt{\mathcal{L}}) V_r^{1/2-1/p} \|_{2 \rightarrow p}. \end{aligned}$$

Claim 1 together with a duality argument easily implies

$$\sup_{r>0} \|(1 + r^2 \mathcal{L})\Phi(r\sqrt{\mathcal{L}})V_r^{1/2-1/p}\|_{2 \rightarrow p} < \infty,$$

while  $(G_p)$  implies that

$$\|\nabla(1 + r^2 \mathcal{L})^{-1}\|_{p \rightarrow p} \leq C \int_0^\infty \|\nabla e^{-t(1+r^2)\mathcal{L}}\|_{p \rightarrow p} dt \leq \frac{C}{r},$$

these two estimate proves the second claim.

**Step 2. Completion of the proof.**

Suppose first that  $u \in W^{1,2}(3B)$ ,  $B = B(x_0, r)$ , satisfies  $\mathcal{L}u = 0$  in  $3B$ . By Claim 2 and the validity of  $(G_{p_0})$ , we then have

$$\|rV_r^{1-1/p_0} |\nabla \Phi(r\sqrt{\mathcal{L}})^2(u\chi_{3B})(\cdot)|\|_{p_0} \leq C\|u\|_{L^1(3B)}.$$

The doubling condition together with Lemma 4.6 implies that

$$\|\nabla u\|_{L^p(B)} \leq \frac{1}{rV_r(x_0)^{1-1/p_0}} \|rV_r^{1-1/p_0} |\nabla \Phi(r\sqrt{\mathcal{L}})^2(u\chi_{3B})(\cdot)|\|_{p_0} \leq C \frac{1}{rV_r(x_0)^{1-1/p_0}} \|u\|_{L^1(3B)},$$

i.e.,

$$\left( \int_B |\nabla u|^{p_0} d\mu \right)^{1/p_0} \leq \frac{C}{r} \int_{3B} |u| d\mu.$$

Finally following the same argument as in **Step 4** of proof of Theorem 3.5, we can see that  $(RH_{p_0})$  holds, which completes the proof.  $\square$

**Remark 4.10.** (i) Using Claim 1 from Step 1 and [21, Proposition 4.1.1] one can see that for each  $r > 0$

$$\begin{aligned} \|V_r \Phi(r\sqrt{\mathcal{L}})^2\|_{1 \rightarrow \infty} &= \|V_r \Phi(r\sqrt{\mathcal{L}}) V_r^{-1/2} V_r^{1/2} \Phi(r\sqrt{\mathcal{L}})\|_{1 \rightarrow \infty} \\ &\leq \|V_r \Phi(r\sqrt{\mathcal{L}}) V_r^{-1/2}\|_{2 \rightarrow \infty} \|V_r^{1/2} \Phi(r\sqrt{\mathcal{L}})\|_{1 \rightarrow 2} \\ &\leq \|\Phi(r\sqrt{\mathcal{L}}) V_r^{1/2}\|_{2 \rightarrow \infty} \|V_r^{1/2} \Phi(r\sqrt{\mathcal{L}})\|_{1 \rightarrow 2} \\ &\leq C. \end{aligned}$$

This together with Lemma 4.6 then gives a simple proof of Proposition 2.1.

We remark that the above proof indeed works also in the local settings in Section 5. However, it is not easy to check this fact, since one also needs to check arguments from [21]. In what follows, we present a more self-contained but more complicated proof of Theorem 4.9, which can be easily generalized to the local settings.

Recall that  $(UE)$  is equivalent to the Sobolev inequality  $(LS_q)$ , for any  $q \in (2, \infty]$  satisfying  $\frac{q-2}{q} < \frac{2}{Q}$ , where  $Q$  is in  $(D_Q)$ . Fix a  $q \in (2, \infty)$  such that  $\frac{q-2}{q} < \frac{2}{Q}$  below. Let  $\tilde{Q} := \frac{2q}{q-2}$ .

Notice that  $(LS_q)$  is equivalent to the following Sobolev inequality  $(S_{q,2})$ ,

$$(S_{q,2}) \quad \left( \int_B |f|^q d\mu \right)^{1/q} \leq C_S r \left( \int_B |\nabla f|^2 d\mu \right)^{1/2},$$

see [21, 57]. This further implies via using  $(S_{q,2})$  and the Hölder inequality that, for  $f \in W_0^{1,p}(B)$ ,  $p \in (2, \tilde{Q})$ , it holds with  $p^* := \frac{\tilde{Q}p}{\tilde{Q}-p}$  that

$$(S_{p^*,p}) \quad \left( \int_B |f|^{p^*} d\mu \right)^{1/p^*} \leq C_S r \left( \int_B |\nabla f|^p d\mu \right)^{1/p}.$$

*Second proof of Theorem 4.9. Step 1. Applying the reproducing formula.*

Suppose that  $u \in W^{1,2}(8B)$ ,  $B = B(x_0, r)$ , satisfies  $\mathcal{L}u = 0$  in  $8B$ . Let  $\psi$  be a Lipschitz cut-off function such that  $\psi = 1$  on  $3B$ ,  $\psi = 0$  outside  $4B$ .

Let  $\Phi \in \mathcal{S}(\mathbb{R})$  be an even function whose Fourier transform  $\hat{\Phi}$  satisfies  $\text{supp } \hat{\Phi} \subset [-1, 1]$  and  $\Phi(0) = 1$ . By Corollary 4.8, we see that  $u = \Phi(r\sqrt{\mathcal{L}})(u\psi)$  in  $W^{1,2}(B)$ .

**Step 2. Claim:**

$$\left( \int_{B(x_0, r)} |\nabla u|^{p_0} d\mu \right)^{1/p_0} \leq \frac{C}{r} \int_{8B} |u| d\mu.$$

For simplicity, in this step of the proof, we set  $v_0 := \Phi(r\sqrt{\mathcal{L}})(u\psi)$ , and define, for  $j \in \mathbb{N}$ ,  $v_j := (r^2 \mathcal{L})^j \Phi(r\sqrt{\mathcal{L}})(u\psi)$ . Then by (4.5), we see that  $\text{supp } v_j \subset \overline{5B}$  for each  $j \in \mathbb{Z}_+$ . By Remark 4.7 and Lemma 2.14,  $v_j \in W_0^{1,2}(5B) \cap \mathcal{D}(\mathcal{L})$  for  $j = 0, 1, \dots$ .

Let  $k_0$  be the smallest natural number such that  $\tilde{Q} \leq 2k_0$ . For each  $k \in \mathbb{N}$  satisfying  $k < k_0$ , let  $p_k := \frac{2\tilde{Q}}{\tilde{Q}-2k}$ , and  $p_{k_0} := \infty$ . Notice that,  $p_1 = q$ , where  $q$  is as in  $(S_{q,2})$  above, and for  $k < k_0$ ,  $p_k = (p_{k-1})^* = \frac{\tilde{Q}p_{k-1}}{\tilde{Q}-p_{k-1}}$ .

Notice that from the assumption of  $L^{p_0}$ -boundedness of  $|\nabla H_t|$ , and the natural  $L^2$ -boundedness of  $|\nabla H_t|$  and an interpolation, it follows that

$$(4.10) \quad ||| \nabla H_t |||_{p \rightarrow p} \leq \frac{C}{\sqrt{t}}$$

for each  $p \in [2, p_0]$ .

**Case 1:**  $p_0 \in (2, p_1]$ .

Recall that  $v_j \in W_0^{1,2}(5B)$  and  $v_j \in \mathcal{D}(\mathcal{L})$  for  $j = 0, 1, \dots$ . By the spectral theory

$$(4.11) \quad \sum_{j=0}^{k_0-1} \|\nabla v_j\|_2 \leq \sum_{j=0}^{k_0-1} \left\| \sqrt{\mathcal{L}}(r^2 \mathcal{L})^j \Phi(r\sqrt{\mathcal{L}})(u\psi) \right\|_2 \leq \frac{C}{r} \|u\psi\|_2,$$

which, together with the Sobolev inequality  $(S_{q,2})$ , implies that

$$(4.12) \quad \sum_{j=0}^{k_0-1} \|v_j\|_{p_1} \leq \frac{Cr}{V(x_0, r)^{1/\tilde{Q}}} \sum_{j=0}^{k_0-1} \|\nabla v_j\|_2 \leq \frac{C}{V(x_0, r)^{1/\tilde{Q}}} \|u\psi\|_2.$$



We can now estimate  $|\nabla\Phi(r\sqrt{\mathcal{L}})(u\psi)| = |\nabla v_0|$ . By (4.10), (4.12) and the fact that  $\text{supp } v_j \subset \overline{5B}$  for each  $j \in \mathbb{Z}_+$ , we see that for each  $p \in (2, p_1] \cap (2, p_0]$

$$\begin{aligned} \left( \int_X |\nabla v_0|^p d\mu \right)^{1/p} &= \left( \int_X |\nabla(1+r^2\mathcal{L})^{-1}(1+r^2\mathcal{L})v_0|^p d\mu \right)^{1/p} \leq C \int_0^\infty \left\| \nabla e^{-t(1+r^2\mathcal{L})}(v_0 + v_1) \right\|_p dt \\ (4.13) \quad &\leq C \int_0^\infty \frac{e^{-t}}{(tr^2)^{1/2}} \|v_0 + v_1\|_p dt \leq \frac{C}{r} V(x_0, r)^{1/p-1/2} \|u\psi\|_{L^2(4B)}, \end{aligned}$$

and similarly for each  $j = 1, \dots, k_0 - 2$ ,

$$\begin{aligned} \left( \int_X |\nabla v_j|^p d\mu \right)^{1/p} &= \left( \int_X |\nabla(1+r^2\mathcal{L})^{-1}(1+r^2\mathcal{L})v_j|^p d\mu \right)^{1/p} \\ &\leq C \int_0^\infty \left\| \nabla e^{-t(1+r^2\mathcal{L})}(v_j + v_{j+1}) \right\|_p dt \\ (4.14) \quad &\leq \frac{C}{r} V(x_0, r)^{1/p-1/2} \|u\psi\|_{L^2(4B)}. \end{aligned}$$

Combining (4.13) with Proposition 2.1, we find that

$$\left( \int_{B(x_0, r)} |\nabla v_0|^p d\mu \right)^{1/p} \leq \frac{C}{r} \frac{1}{V(x_0, r)^{1/2}} \|u\|_{L^2(4B)} \leq \frac{C}{r} \int_{8B} |u| d\mu.$$

From this and **Step 1**, we can conclude that

$$\left( \int_{B(x_0, r)} |\nabla u|^{p_0} d\mu \right)^{1/p_0} = \left( \int_{B(x_0, r)} |\nabla v_0|^{p_0} d\mu \right)^{1/p_0} \leq \frac{C}{r} \int_{8B} |u| d\mu.$$

**Case 2:**  $p_0 \in (p_1, \infty)$ .

This case follows by iterating the proof of **Case 1**. Observe that there exists  $k < k_0$ , such that  $(p_0)_* \in (2, p_k]$ , where  $(p_0)_* := \frac{\tilde{Q}p_0}{\tilde{Q}+p_0}$  is the lower Sobolev conjugate of  $p_0$ .

Notice that now both (4.13) and (4.14) work for  $p_1 = \frac{2\tilde{Q}}{\tilde{Q}-2}$ . Applying this and the Sobolev inequality  $(S_{p_2, p_1})$ ,  $p_2 = (p_1)^*$ , and following the proof of (4.12), we can conclude that

$$(4.15) \quad \sum_{j=0}^{k_0-2} \|v_j\|_{p_2} \leq \frac{Cr}{V(x_0, r)^{1/\tilde{Q}}} \left( \sum_{j=0}^{k_0-2} \|\nabla v_j\|_{p_1} \right) \leq \frac{C}{V(x_0, r)^{2/\tilde{Q}}} \|u\psi\|_2.$$

For each  $p \in (p_1, p_2] \cap (2, p_0]$ , by (4.10) and (4.15), similarly as (4.13), we see that

$$(4.16) \quad \sum_{j=0}^{k_0-3} \|\nabla v_j\|_p \leq \frac{C}{r} V(x_0, r)^{1/p-1/2} \|u\psi\|_{L^2(4B)}.$$

Combining (4.16), Proposition 2.1 and **Step 1**, we conclude that for each  $p \in (p_1, p_2] \cap (2, p_0]$ ,

$$\left( \int_{B(x_0, r)} |\nabla u|^p d\mu \right)^{1/p} = \left( \int_{B(x_0, r)} |\nabla v_0|^p d\mu \right)^{1/p} \leq \frac{C}{r} \int_{8B} |u| d\mu.$$

Iterating this argument  $k_0 - 2$  times more, we can finally deduce that for  $p_0 \in (p_1, \infty)$ ,

$$\left( \int_{B(x_0, r)} |\nabla u|^{p_0} d\mu \right)^{1/p_0} \leq \frac{C}{r} \int_{8B} |u| d\mu.$$

**Case 3:**  $p_0 = \infty$ .

Since  $(G_\infty)$  implies  $(G_p)$  for all  $p \in (2, \infty)$ , the conclusion of **Case 2** works here. Let us fix a  $p \in (\tilde{Q}, \infty) \cap (Q, \infty)$ . From **Case 2**, similar to (4.15), we see that

$$(4.17) \quad \|v_0\|_p + \|v_1\|_p \leq CV(x_0, r)^{1/p-1/2} \|u\psi\|_2.$$

Noticing that  $\text{supp } v_0, \text{supp } v_1 \subset \overline{5B}$ , and applying (UE) and (4.4), we find that for all  $t > 0$  and  $x \in X$  that,

$$(4.18) \quad \begin{aligned} |H_t v_0(x)| &\leq \int_{5B} \frac{C}{V(y, \sqrt{t})} \exp \left\{ -c \frac{d^2(x, y)}{t} \right\} |v_0(y)| d\mu(y) \\ &\leq \frac{C(r + \sqrt{t})^{Q/p}}{t^{\frac{Q}{2p}} V(x_0, 5r + \sqrt{t})^{1/p}} \|v_0\|_p \leq \frac{C(r + \sqrt{t})^{Q/p}}{t^{\frac{Q}{2p}} V(x_0, r)^{1/p}} \|v_0\|_p \leq \frac{C(r + \sqrt{t})^{Q/p}}{t^{\frac{Q}{2p}} V(x_0, r)^{1/2}} \|u\psi\|_2, \end{aligned}$$

and

$$(4.19) \quad |H_t v_1(x)| \leq \frac{C(r + \sqrt{t})^{Q/p}}{t^{\frac{Q}{2p}} V(x_0, r)^{1/2}} \|u\psi\|_2.$$

Combining these two estimates with Proposition 2.1, we can conclude that

$$(4.20) \quad \begin{aligned} \|\nabla v_0\|_\infty &= \left\| \nabla (1 + r^2 \mathcal{L})^{-1} (1 + r^2 \mathcal{L}) v_0 \right\|_\infty \\ &\leq C \int_0^\infty \left\| \nabla e^{-t - \frac{t^2}{2}} \mathcal{L} [e^{-\frac{t^2}{2}} \mathcal{L} (v_0 + v_1)] \right\|_\infty dt \\ &\leq C \int_0^\infty \left\| \nabla e^{-t - \frac{t^2}{2}} \mathcal{L} \right\|_{\infty \rightarrow \infty} \frac{(r + r\sqrt{t})^{Q/p}}{(r^2 t)^{\frac{Q}{2p}} V(x_0, r)^{1/2}} \|u\psi\|_2 dt \\ &\leq C \int_0^\infty \frac{e^{-t}}{(tr^2)^{1/2}} \frac{(1 + \sqrt{t})^{Q/p}}{t^{\frac{Q}{2p}} V(x_0, r)^{1/2}} \|u\psi\|_2 dt \\ &\leq \frac{C}{r} V(x_0, r)^{-1/2} \|u\|_{L^2(4B)} \leq \frac{C}{r} \int_{8B} |u| d\mu. \end{aligned}$$

Once more, using **Step 1**, we finally deduce that

$$\|\nabla u\|_{L^\infty(B)} = \|\nabla v_0\|_{L^\infty(B)} \leq \frac{C}{r} \int_{8B} |u| d\mu.$$

The above three cases complete the proof of **Step 2**.

**Step 3.** *Completion of the proof.*

Using **Step 2** and following the same argument as in **Step 4** of proof of Theorem 3.5, we can see that  $(RH_{p_0})$  holds, as desired.  $\square$

We can now finish the proofs of Theorem 1.2 and Theorem 1.6, and their corollaries.

*Proof of Theorem 1.2.*  $(RH_\infty) \implies (GLY_\infty)$  is contained in Proposition 4.1,  $(GLY_\infty) \implies (G_\infty)$  is straightforward and is contained in Proposition 4.2 (see [7, p.919]), and  $(G_\infty) \implies (RH_\infty)$  is contained in Theorem 4.9.

$(GBE) \implies (GLY_\infty)$  follows from [7, Lemma 3.3] whose proof requires only  $(D)$  and  $(UE)$ . Notice that  $(GLY_\infty)$  together with  $(UE)$  implies  $(LY)$ , and therefore  $(P_2)$ ; see [15, Theorem 3.4]. Using  $(D)$  and  $(P_2)$ ,  $(GLY_\infty) \implies (GBE)$  then also follows from [7, Lemma 3.3]; see also the proof of Theorem 5.5 below.  $\square$

*Proof of Corollary 1.3.* Note that  $(P_2)$  implies  $(P_\infty)$  and  $(LY)$ , in particular  $(P_{\infty, \text{loc}})$  and  $(UE)$ .  $\square$

*Proof of Corollary 1.4.* If  $(X, d, \mu)$  is a Riemannian manifold, then for any locally smooth function  $v$  with bounded gradient  $\nabla v$  on a ball  $B$ ,  $B = B(x_0, r)$ , it holds

$$\int_B |v - v_B| d\mu \leq \int_B \int_B |v(x) - v(y)| d\mu(x) d\mu(y) \leq Cr \|\nabla v\|_{L^\infty(B)}.$$

Since harmonic function is locally smooth on a Riemannian manifold, this together with the assumption  $(RH_\infty)$  implies Lemma 3.3 holds. Therefore,  $(D)$  and  $(UE)$  are enough to guarantee  $(RH_\infty) \implies (GLY_\infty)$  if  $(X, d, \mu)$  is a Riemannian manifold.

The implications  $(GLY_\infty) \implies (G_\infty)$  and  $(G_\infty) \implies (RH_\infty)$  are contained in Proposition 4.2 and Theorem 4.9, respectively, requiring only  $(D)$  and  $(UE)$ .

$(GBE) \implies (GLY_\infty)$  is straightforward; see [7, Lemma 3.3]. On the other hand, since under  $(D)$  and  $(GLY_\infty)$ ,  $(P_2)$  holds by [35, Corollary 2.2] (see also [15, Theorem 3.4]), and one can apply [7, Lemma 3.3] to see that  $(GLY_\infty) \implies (GBE)$ .  $\square$

*Proof of Theorem 1.6.*  $(RH_p) \implies (GLY_p)$  is contained in Proposition 4.3,  $(GLY_p) \implies (G_p)$  is explained in Proposition 4.4, and  $(G_p) \implies (RH_p)$  is contained in Theorem 4.9.  $\square$

*Proof of Corollary 1.7.* The corollary holds, since  $(P_2)$  implies  $(P_{2, \text{loc}})$  and  $(UE)$ .  $\square$

## 5 Localisation

In this section, we give the localisations of our main results. Notice that our results in this section also covers the compact cases, which were not addressed in the introduction.

Let  $(X, d, \mu, \mathcal{E})$  be a metric measure Dirichlet space endowed with a “carré du champ”. We shall say that  $(X, d, \mu, \mathcal{E})$  satisfies the local volume doubling property  $(EV)$ , if for each  $\theta > 1$ ,

$$(EV) \quad V(x, \theta r) \leq C e^{c\theta} V(x, r).$$

Notice that  $(EV)$  implies that, for all  $r_0 > 0$  there exists a constant  $C_D(r_0)$ , such that

$$(D_{\text{loc}}) \quad V(x, 2r) \leq C_D(r_0) V(x, r)$$

for all  $0 < r < r_0$  and  $x \in X$ . This implies that there exists  $Q > 0$ , and for all  $r_0 > 0$  there exists  $C_Q(r_0) > 0$  such that

$$(D_{Q,\text{loc}}) \quad V(x, R) \leq C_Q(r_0) \left( \frac{R}{r} \right)^Q V(x, r)$$

for all  $0 < r < R \leq r_0$  and  $x \in X$ . Apparently  $Q$  depends on  $r_0$ , but small calculations using  $(D_{\text{loc}})$  show that one can find a  $Q$  that works for all  $r_0 > 0$ . Notice that  $(D_{\text{loc}})$  holds if and only  $(D_{Q,\text{loc}})$  holds for some  $Q > 0$ . Similarly as in the global version, we may and do assume  $Q > 1$ . Notice that, under  $(EV)$  the heat semigroup is conservative, i.e.  $H_t 1 = 1$  for all  $t > 0$ ; see [99].

We say that a local upper Gaussian bound holds for the heat kernel  $h_t$ , if for  $t \in (0, 1]$  and all  $x, y \in X$ ,

$$(UE_{\text{loc}}) \quad h_t(x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp \left\{ -c \frac{d^2(x, y)}{t} \right\}.$$

Typical examples of such spaces  $(X, d, \mu, \mathcal{E})$  include Riemannian manifolds with Ricci curvature bounded from below, and their images under bi-Lipschitz mappings.

**Definition 5.1.** Let  $(X, d, \mu, \mathcal{E})$  be a metric measure Dirichlet space endowed with a “carré du champ” and let  $p \in (2, \infty)$ . We say that the local reverse Hölder inequality for gradients of harmonic functions holds if, for each  $r_0 > 1$ , there exists  $C(r_0) > 0$  such that for every  $u \in W^{1,2}(2B)$ ,  $B = B(x_0, r)$  with  $r < r_0$ , satisfying  $\mathcal{L}u = 0$  in  $2B$ ,

$$(RH_{p,\text{loc}}) \quad \left( \int_B |\nabla u|^p d\mu \right)^{1/p} \leq \frac{C(r_0)}{r} \int_{2B} |u| d\mu.$$

For  $p = \infty$ , we replace the left-hand-side with  $\|\nabla u\|_{L^\infty(B)}$ .

We have the following local version of Theorem 1.2 that especially covers the case of compact spaces, left over by the main results Theorem 1.2 and Theorem 1.6.

**Theorem 5.2.** Assume that the metric measure Dirichlet space  $(X, d, \mu, \mathcal{E})$  satisfies  $(EV)$ ,  $(UE_{\text{loc}})$  and  $(P_{\infty,\text{loc}})$ . Then the following statements are equivalent:

- (i)  $(RH_{\infty,\text{loc}})$  holds.
- (ii) There exist  $C, c > 0$  such that

$$(GLY_{\infty,\text{loc}}) \quad |\nabla_x h_t(x, y)| \leq \frac{C}{\sqrt{t}V(y, \sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{ct} \right\}$$

for every  $t \in (0, 1]$  and a.e.  $x, y \in X$ .

- (iii) The gradient of the heat semigroup  $|\nabla H_t|$  is bounded on  $L^\infty(X)$  for each  $t > 0$  with

$$(G_{\infty,\text{loc}}) \quad \|\nabla H_t\|_{\infty \rightarrow \infty} \leq \frac{C}{\sqrt{t} \wedge 1}.$$

We were not able to include a generalized Bakry-Émery condition above, since our Poincaré inequality is rather weak. We will show in Theorem 5.5 below, that if one has a stronger local  $L^2$ -Poincaré inequality, then the above three conditions is equivalent to a generalized Bakry-Émery condition, i.e., there exist  $C, c > 0$  such that for each  $t \in (0, 1]$  it holds for each  $f \in W^{1,2}(X)$  and a.e.  $x \in X$  that

$$(GBE_{\text{loc}}) \quad |\nabla H_t f(x)|^2 \leq C H_{ct}(|\nabla f|^2)(x).$$

Note  $(GBE_{\text{loc}})$  implies for all  $t > 0$  that  $|\nabla H_t f(x)|^2 \leq C e^{\tilde{C}t} H_{ct}(|\nabla f|^2)(x)$ .

Regarding the local  $L^p$ -version, we have the following local version of Theorem 1.6.

**Theorem 5.3.** *Let  $p \in (2, \infty)$ . Assume that the metric measure Dirichlet space  $(X, d, \mu, \mathcal{E})$  satisfies  $(EV)$  and  $(P_{2,\text{loc}})$ . Then the following statements are equivalent:*

- (i)  $(RH_{p,\text{loc}})$  holds.
- (ii) There exists  $\gamma > 0$  such that

$$(GLY_{p,\text{loc}}) \quad \int_X |\nabla_x h_t(x, y)|^p \exp\{\gamma d^2(x, y)/t\} d\mu(x) \leq \frac{C}{t^{p/2} V(y, \sqrt{t})^{p-1}}$$

for every  $t \in (0, 1]$  and a.e.  $y \in X$ .

- (iii) The gradient of the heat semigroup  $|\nabla H_t|$  is bounded on  $L^p(X, \mu)$  for each  $t > 0$  with

$$(G_{p,\text{loc}}) \quad |||\nabla H_t|||_{p \rightarrow p} \leq \frac{C}{\sqrt{t} \wedge 1}.$$

**Remark 5.4.** In the above theorem, the condition  $(P_{2,\text{loc}})$  can be replaced by the weaker condition  $(UE_{\text{loc}})$  together with  $(P_{p,\text{loc}})$ . Arguing similar to [15, Theorem 6.1] and [16, Corollary 3.8], one can see that,  $(UE_{\text{loc}})$  and  $(P_{p,\text{loc}})$  together with  $(RH_{p,\text{loc}})$  or  $(G_{p,\text{loc}})$  imply  $(P_{2,\text{loc}})$ .

Notice that under  $(EV)$  and the local  $L^2$ -Poincaré inequality, one has the local Li-Yau estimate,

$$(LY_{\text{loc}}) \quad \frac{C^{-1}}{V(x, \sqrt{t})} \exp\left\{-\frac{d^2(x, y)}{ct}\right\} \leq h_t(x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp\left\{-c\frac{d^2(x, y)}{t}\right\},$$

for  $t \in (0, 1]$  and all  $x, y \in X$ . Using the Cauchy transform, one has that,  $(UE_{\text{loc}})$  and hence  $(LY_{\text{loc}})$ , implies the following estimate for the time gradient of heat kernel,

$$(5.1) \quad \left| \frac{\partial h_t}{\partial t}(x, y) \right| \leq \frac{C}{tV(y, \sqrt{t})} \exp\left\{-\frac{d^2(x, y)}{ct}\right\},$$

for  $t \leq 1$ ; see [100]. Since  $(X, d, \mu)$  is a geodesic space, we have that for all  $x, y \in X$ , there is a curve  $\gamma$  connecting  $x$  and  $y$  and satisfying  $\text{length}(\gamma) = d(x, y)$ . This and the local doubling condition imply that, for each  $t \leq 1$

$$(5.2) \quad V(x, \sqrt{t}) \leq C \exp\left(\frac{Cd(x, y)}{\sqrt{t}}\right) V(y, \sqrt{t});$$

see [69, p.39]. This shows that,  $\frac{\partial h}{\partial t}(x, y)$  satisfies (5.1) with  $V(y, \sqrt{t})$  replaced by  $V(x, \sqrt{t})$ .

Moreover, under  $(D_{\text{loc}})$ ,  $(UE_{\text{loc}})$  is equivalent to a local Sobolev inequality  $(LS_{q, \text{loc}})$  for any  $q \in (2, \infty]$  with  $\frac{q-2}{q}Q < 2$ , where  $Q$  is the dimension from  $(D_{Q, \text{loc}})$ . Here  $(LS_{q, \text{loc}})$  means that, for each  $r_0 > 0$  there exists  $C_{LS}(r_0) > 0$  such that for every ball  $B = B(x, r)$  with  $r < r_0$  and each  $f \in W_0^{1,2}(B)$ ,

$$(LS_{q, \text{loc}}) \quad \left( \int_B |f|^q d\mu \right)^{2/q} \leq C_{LS}(r_0) \left( \int_B |f|^2 d\mu + \frac{r^2}{V(x, r)} \mathcal{E}(f, f) \right).$$

This equivalence is contained in [21, Theorem 1.2.1] by chosen  $v(x, r)$  as  $\min\{V(x, r), V(x, r_0)\}$ . Therefore, under  $(D_{\text{loc}})$  and  $(UE_{\text{loc}})$ , Lemma 2.5 and Lemma 2.8 work for small balls.

As the proofs of Theorem 1.2 and Theorem 1.6 work in the local settings as well, we will sketch proofs for Theorem 5.3 and Theorem 5.2 here.

*Proof of Theorem 5.3. Step 1.*  $(RH_{p, \text{loc}}) \implies (GLY_{p, \text{loc}})$ .

**Step 1.1.** Regularity of solutions to Poisson equations.

Notice that in the case of a local doubling condition together with  $(UE_{\text{loc}})$  and  $(P_{p, \text{loc}})$ , Theorem 3.2 and Theorem 3.5 work for small balls, i.e., for a fixed  $r_0 > 0$ , it holds for all  $r < r_0$  that

$$(5.3) \quad \left( \int_{B(x_0, r)} |\nabla f|^p d\mu \right)^{1/p} \leq \frac{C(r_0)}{r} \left( \int_{B(x_0, 2r)} |f| d\mu + r^2 \left( \int_{B(x_0, 2r)} |g|^q d\mu \right)^{1/q} \right),$$

where  $\mathcal{L}f = g$  in  $2B$  and  $g \in L^\infty(2B)$ ,  $p_Q < q \leq p$  with  $1/q - 1/p < 1/Q$  and  $C$  depends on  $r_0$ .

**Step 1.2.** *Claim:* There exists  $\gamma > 0$  such that for each  $t \in (0, 1]$

$$\int_X |\nabla_x h_t(x, y)|^p \exp\{\gamma d^2(x, y)/t\} d\mu(x) \leq \frac{C(r_0)}{t^{p/2} V(y, \sqrt{t})^{p-1}}.$$

Decompose the space  $X$  via  $B = B(y, 2\sqrt{t})$  and  $U_k(B) := B(y, 2^{k+1}\sqrt{t}) \setminus B(y, 2^k\sqrt{t})$ ,  $k \geq 1$ . Using (5.3) we see that

$$(5.4) \quad \begin{aligned} & \| |\nabla_x h_t(\cdot, y)| \|_{L^p(B)} \\ & \leq \frac{CV(y, 4\sqrt{t})^{1/p}}{\sqrt{t}} \left( \int_{B(y, 4\sqrt{t})} |h_t(x, y)| d\mu(x) + t \left( \int_{B(y, 4\sqrt{t})} \left| \frac{\partial}{\partial t} h_t(x, y) \right|^p d\mu(x) \right)^{1/p} \right) \\ & \leq \frac{C}{\sqrt{t}} V(y, \sqrt{t})^{1/p-1}. \end{aligned}$$

Let  $k \geq 1$  and consider  $U_k(B)$ . Let  $\{B_{k,j} = B(x_{k,j}, \sqrt{t}/2)\}_j$  be a maximal set of pairwise disjoint balls with radius  $2^{-1}\sqrt{t}$  in  $B(y, 2^{k+1}\sqrt{t})$ . Then it is easy to see that  $B(y, 2^{k+1}\sqrt{t}) \subset \cup_j B(x_{k,j}, \sqrt{t})$  and

$$\sum_j \chi_{4B_{k,j}}(x) \leq C(C_D(1)).$$

Therefore, by applying (5.3), the local doubling condition and  $(UE_{\text{loc}})$ , (5.1) we can conclude that

$$\begin{aligned}
& \int_{U_k(B)} |\nabla_x h_t(x, y)|^p d\mu(x) \\
& \leq \sum_{j: 2B_{k,j} \cap U_k(B) \neq \emptyset} \int_{2B_{k,j}} |\nabla_x h_t(x, y)|^p d\mu(x) \\
& \leq \sum_{j: 2B_{k,j} \cap U_k(B) \neq \emptyset} \frac{C\mu(4B_{k,j})}{t^{p/2}} \left( \int_{4B_{k,j}} |h_t(x, y)| d\mu(x) + t \left( \int_{4B_{k,j}} \left| \frac{\partial}{\partial t} h_t(x, y) \right|^p d\mu(x) \right)^{1/p} \right)^p \\
& \leq \sum_{j: 2B_{k,j} \cap U_k(B) \neq \emptyset} \frac{C\mu(B_{k,j})}{t^{p/2}} V(y, \sqrt{t})^{-p} \exp \left\{ \frac{-c2^{2k}t}{t} \right\} \\
& \leq \sum_{j: 2B_{k,j} \cap U_k(B) \neq \emptyset} \frac{C}{t^{p/2}} V(y, \sqrt{t})^{1-p} \exp \left\{ -c2^{2k-1} \right\} \int_{B_{k,j}} V(y, \sqrt{t})^{-1} \exp \left\{ \frac{-cd^2(x, y)}{2t} \right\} d\mu(x) \\
& \leq C \frac{C}{t^{p/2}} V(y, \sqrt{t})^{1-p} \exp \left\{ -c2^{2k-1} \right\} \int_X V(y, \sqrt{t})^{-1} \exp \left\{ \frac{-cd^2(x, y)}{2t} \right\} d\mu(x) \\
& \leq C \frac{C}{t^{p/2}} V(y, \sqrt{t})^{1-p} \exp \left\{ -c2^{2k-1} \right\}.
\end{aligned}$$

Above in the last inequality, we used the following estimate, which needs  $(EV)$ ,

$$\begin{aligned}
\int_X V(y, \sqrt{t})^{-1} \exp \left\{ \frac{-cd^2(x, y)}{2t} \right\} d\mu(x) & \leq \int_{B(y, 2\sqrt{t})} V(y, \sqrt{t})^{-1} \exp \left\{ \frac{-cd^2(x, y)}{2t} \right\} d\mu(x) \\
& \quad + \sum_{j=1}^{\infty} \int_{B(y, 2^{j+1}\sqrt{t}) \setminus B(y, 2^j\sqrt{t})} V(y, \sqrt{t})^{-1} \exp \left\{ \frac{-cd^2(x, y)}{2t} \right\} d\mu(x) \\
& \leq C + C \sum_{j=1}^{\infty} e^{-c2^{2j}} \frac{V(y, 2^{j+1}\sqrt{t})}{V(y, \sqrt{t})} \\
(5.5) \quad & \leq C + C \sum_{j=1}^{\infty} e^{-c2^{2j}} e^{c12^j} \leq C.
\end{aligned}$$

From this and the estimate (5.4), we deduce that, there exists  $\gamma > 0$ , such that for each  $t \in (0, 1]$

$$(GLY_{p, \text{loc}}) \quad \int_X |\nabla_x h_t(x, y)|^p \exp \left\{ \gamma d^2(x, y)/t \right\} d\mu(x) \leq \frac{C(\mu)}{t^{p/2} V(y, \sqrt{t})^{p-1}},$$

as desired.

**Step 2.**  $(GLY_{p, \text{loc}}) \implies (G_{p, \text{loc}})$ .

By using inequality  $(GLY_{p, \text{loc}})$ ,  $(UE_{\text{loc}})$  and (5.5), we deduce that for each  $f \in L^p(X, \mu)$ ,

$$\| \nabla H_t f \|_p^p \leq \int_X \left( \int_X |\nabla_x h_t(x, y)| |f(y)| d\mu(y) \right)^p d\mu(x)$$

$$\begin{aligned}
&\leq \int_X \left( \int_X |\nabla_x h_t(x, y)|^p \exp \{ \gamma d^2(x, y)/t \} [V(y, \sqrt{t})]^{p-1} |f(y)|^p d\mu(y) \right) \\
&\quad \times \left( \int_X \frac{1}{V(y, \sqrt{t})} \exp \left\{ -\frac{p' \gamma d^2(x, y)}{pt} \right\} d\mu(y) \right)^{1/p'} d\mu(x) \\
&\leq C \int_X \int_X |\nabla_x h_t(x, y)|^p \exp \{ \gamma d^2(x, y)/t \} [V(y, \sqrt{t})]^{p-1} |f(y)|^p d\mu(y) d\mu(x) \\
&\leq \frac{C}{t^{p/2}} \|f\|_p^p.
\end{aligned}$$

This yields

$$\|\nabla H_t\|_{p \rightarrow p} \leq C t^{-1/2}$$

for  $t \leq 1$ . For  $t > 1$ , by writing  $H_t f = H_1(H_{t-1}f)$ , we conclude that

$$\|\nabla H_t\|_{p \rightarrow p} \leq \|\nabla H_1\|_{p \rightarrow p} \|H_{t-1}\|_{p \rightarrow p} \leq C,$$

as desired.

**Step 3.**  $(G_{p, \text{loc}}) \implies (RH_{p, \text{loc}})$ .

The proof of  $(G_{p, \text{loc}}) \implies (RH_{p, \text{loc}})$  is identical to the second proof of Theorem 4.9. Indeed, it is based on (i)  $L^2$ -boundedness of the spectral operators, (ii) a reproducing for harmonic functions, (iii)  $(UE)$  or equivalently, a Sobolev inequality, and (iv) boundedness of the operator  $|\nabla(1+r^2\mathcal{L})^{-1}|$ .

By Lemma 2.14 we see that  $L^2$ -boundedness of the spectral operators holds in the local setting.

Since our reproducing formula is a local property, it is easy to see that Lemma 4.6 and Corollary 4.8 work in the local setting.

A local Sobolev inequality  $(LS_{q, \text{loc}})$  holds under  $(UE_{\text{loc}})$ , which is enough for our purpose, as we assume that the radii of balls are small.

Finally, let us check the boundedness of the operator  $|\nabla(1+r^2\mathcal{L})^{-1}|$ , where  $r \leq 1$ . For each  $f \in L^p(X, \mu)$ , we have

$$\|\nabla(1+r^2\mathcal{L})^{-1}f\|_p \leq C \int_0^\infty \|\nabla e^{-t(1+r^2\mathcal{L})}f\|_p dt \leq C\|f\|_p \int_0^\infty \frac{e^{-t}}{[(tr^2) \wedge 1]^{1/2}} dt \leq \frac{C}{r}\|f\|_p.$$

The remaining proofs are the same as those for Theorem 4.9. The proof is complete.  $\square$

The proof of Theorem 5.2 can be carried out similarly.

*Proof of Theorem 5.2.*  $(RH_{\infty, \text{loc}}) \implies (GLY_{\infty, \text{loc}})$ . The fact that  $(RH_{\infty, \text{loc}})$  implies  $(GLY_{\infty, \text{loc}})$  follows by the argument of proof of Proposition 4.1, by using a gradient estimate for Poisson equations locally.

$(GLY_{\infty, \text{loc}}) \implies (G_{\infty, \text{loc}})$ . The fact that  $(GLY_{\infty, \text{loc}})$  implies

$$\|\nabla H_t\|_{\infty \rightarrow \infty} \leq \frac{C}{\sqrt{t} \wedge 1}$$



follows by using  $(UE_{\text{loc}})$  and  $\|H_t\|_{\infty \rightarrow \infty} \leq 1$ .

$(G_{\infty, \text{loc}}) \implies (RH_{\infty, \text{loc}})$ . The proof is identical to that of Theorem 4.9, as indicated in **Step 3** of the proof of Theorem 5.3.  $\square$

Next let us strengthen the assumptions on local Poincaré inequalities. We say that  $(X, d, \mu, \mathcal{E})$  supports a *strengthened* local  $L^2$ -Poincaré inequality if there exists  $C, C_P > 0$ , such that for all  $r > 0$ , it holds for all Lipschitz functions  $f$  on any  $B = B(x_0, r)$ , that

$$(5.6) \quad \int_{B(x_0, r)} |f - f_B| d\mu \leq C e^{C_P r} r \left( \int_{B(x_0, r)} |\nabla f|^2 d\mu \right)^{1/2},$$

Notice that (5.6) hold on Riemannian metric measure spaces with lower Ricci curvature bounds, i.e.  $RCD^*(K, N)$  spaces with  $K < 0$  and  $N < \infty$ ; see [44, 103, 90].

**Theorem 5.5.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies (EV) and a strengthened local  $L^2$ -Poincaré inequality. Then the following statements are equivalent.*

(i)  $(RH_{\infty, \text{loc}})$  holds.

(ii) There exist  $C, c > 0$  such that for each  $t \in (0, 1]$  it holds for a.e.  $x, y \in X$

$$(GLY_{\infty, \text{loc}}) \quad |\nabla_x h_t(x, y)| \leq \frac{C}{\sqrt{t}V(y, \sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{ct} \right\}.$$

(iii) The gradient of the heat semigroup  $|\nabla H_t|$  is bounded on  $L^\infty(X)$  for each  $t > 0$  with

$$(G_{\infty, \text{loc}}) \quad \|\nabla H_t\|_{\infty \rightarrow \infty} \leq \frac{C}{\sqrt{t} \wedge 1}.$$

(iv) There exist  $C, c > 0$  such that for each  $t \in (0, 1]$  it holds for each  $f \in W^{1,2}(X)$  and a.e.  $x \in X$  that

$$(GBE_{\text{loc}}) \quad |\nabla H_t f(x)|^2 \leq C H_{ct}(|\nabla f|^2)(x).$$

*Proof.* The equivalences of (i), (ii) and (iii) are contained in Theorem 5.2, since (5.6) implies  $(UE_{\text{loc}})$  and  $(P_{\infty, \text{loc}})$ .

Let us prove the equivalence of (ii) and (iv). The proof is similar to the proof of [7, Lemma 3.3].

**Step 1.**  $(ii) \implies (iv)$ .

Let  $t \in (0, 1]$  and  $f \in W^{1,2}(X)$ . Let  $x \in X$  such that for a.e.  $y \in X$

$$|\nabla_x h_t(x, y)| \leq \frac{C}{\sqrt{t}V(y, \sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{ct} \right\}.$$

Notice that this together with (5.2) implies

$$|\nabla_x h_t(x, y)| \leq \frac{C}{\sqrt{t}V(x, \sqrt{t})} \exp \left\{ -\frac{d^2(x, y)}{ct} \right\},$$

since  $(X, d)$  is a geodesic space. Now, let  $B = B(x, 2\sqrt{t})$ ,  $U_j = 2^{j+1}B \setminus 2^jB$  for each  $j \in \mathbb{N}$ . Write

$$|\nabla H_t f(x)| = |\nabla H_t(f - f_B)(x)| \leq \int_B \frac{C|f(y) - f_B|}{\sqrt{t}V(x, \sqrt{t})} d\mu(y) + \sum_{j \geq 1} \int_{U_j} \frac{Ce^{-c4^j}|f(y) - f_B|}{\sqrt{t}V(x, \sqrt{t})} d\mu(y).$$

Applying the strengthened local Poincaré inequality (5.6), (EV) and  $(LY_{\text{loc}})$ , we see that

$$\int_B \frac{|f(y) - f_B|}{\sqrt{t}V(x, \sqrt{t})} d\mu(y) \leq \frac{C}{V(x, \sqrt{t})^{1/2}} \left( \int_B |\nabla f|^2 d\mu(y) \right)^{1/2} \leq C \left( \int_B h_{\varepsilon t}(x, y) |\nabla f|^2 d\mu(y) \right)^{1/2},$$

and for each  $j \in \mathbb{N}$ ,

$$\begin{aligned} \int_{U_j} \frac{e^{-c4^j}|f(y) - f_B|}{\sqrt{t}V(x, \sqrt{t})} d\mu(y) &\leq e^{-c4^j} \int_{U_j} \frac{|f(y) - f_{2^{j+1}B}| + |f_B - f_{2^{j+1}B}|}{\sqrt{t}V(x, \sqrt{t})} d\mu(y) \\ &\leq \frac{Ce^{-c4^j}}{\sqrt{t}V(x, \sqrt{t})} \frac{\mu(2^{j+1}B)}{\mu(B)} \int_{2^{j+1}B} |f(y) - f_{2^{j+1}B}| d\mu(y) \\ &\leq \frac{C2^j e^{-c4^j}}{V(x, \sqrt{t})} \frac{\mu(2^{j+1}B)^{3/2}}{\mu(B)} e^{C_P 2^{j+1}\sqrt{t}} \left( \int_{2^{j+1}B} |\nabla f(y)|^2 d\mu(y) \right)^{1/2} \\ &\leq \frac{Ce^{-c4^j + \tilde{c}2^j + C_P 2^{j+1}}}{V(x, \sqrt{t})^{1/2}} \left( \int_{2^{j+1}B} |\nabla f(y)|^2 d\mu(y) \right)^{1/2} \\ &\leq Ce^{-c4^j/2 + \tilde{c}2^j + C_P 2^{j+1}} \left( \int_{2^{j+1}B} h_{\varepsilon t}(x, y) |\nabla f(y)|^2 d\mu(y) \right)^{1/2}, \end{aligned}$$

for some  $\varepsilon > 0$  which does not depend on  $t$ . Combining the above two estimates, we conclude that

$$\begin{aligned} |\nabla H_t f(x)| &= |\nabla H_t(f - f_B)(x)| \leq C \left( H_{\varepsilon t}(|\nabla f|^2)(x) \right)^{1/2} + \sum_{j \geq 1} Ce^{-c4^j/2 + \tilde{c}2^j} \left( H_{\varepsilon t}(|\nabla f|^2)(x) \right)^{1/2} \\ &\leq C \left( H_{\varepsilon t}(|\nabla f|^2)(x) \right)^{1/2}, \end{aligned}$$

as desired.

**Step 2.** (iv)  $\implies$  (ii).

For  $t \in (0, 1]$  and a.e.  $x \in X$ , by  $(GBE_{\text{loc}})$ , we have

$$(5.7) \quad |\nabla_x h_{2t}(x, y)|^2 = |\nabla_x H_t(h_t(\cdot, y))(x)|^2 \leq C \int_X h_{\varepsilon t}(x, z) |\nabla_z h_t(z, y)|^2 d\mu(z).$$

Notice that, similarly to **Step 1.2** of proof of Theorem 5.3, by using the Caccioppoli inequality Lemma 2.4 instead of a gradient estimate for the Poisson equation, we can see that there exists  $\gamma > 0$  such that for each  $t \in (0, 1]$

$$\int_X |\nabla_x h_t(x, y)|^2 \exp\{\gamma d^2(x, y)/t\} d\mu(x) \leq \frac{C(r_0)}{tV(y, \sqrt{t})}.$$

This together with (5.7) implies that

$$\begin{aligned} |\nabla_x h_{2t}(x, y)|^2 &\leq C \int_X h_{ct}(x, z) \exp\{-\gamma d^2(z, y)/t\} \exp\{\gamma d^2(z, y)/t\} |\nabla_z h_t(z, y)|^2 d\mu(z) \\ &\leq \frac{C}{t V(y, \sqrt{t})} \sup_{z \in X} \left( \frac{1}{V(x, \sqrt{t})} \exp\{-cd^2(z, x)/t\} \exp\{-\gamma d^2(z, y)/t\} \right) \\ &\leq C \frac{C}{t V(y, \sqrt{t}) V(x, \sqrt{t})} \exp\{-cd^2(y, x)/t\}, \end{aligned}$$

which implies that

$$|\nabla_x h_{2t}(x, y)| \leq \frac{C}{\sqrt{t} V(y, \sqrt{t})} \exp\{-cd^2(y, x)/t\},$$

as desired.  $\square$

## 6 Riesz transforms

In this section we apply our results to the Riesz transform. The following result was essentially proved by Auscher, Coulhon, Duong and Hofmann [7]; see [15]. As we already said,  $(D)$  together with  $(P_2)$  guarantee  $(R_p)$  for all  $p \in (1, 2]$ , see [33].

**Theorem 6.1.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies  $(D)$  and  $(P_2)$ . Let  $p_0 \in (2, \infty)$ . Then the following statements are equivalent:*

- (i)  $(R_p)$  holds for all  $p \in (2, p_0)$ .
- (ii)  $(G_p)$  holds for all  $p \in (2, p_0)$ .

First we record the open-ended character of condition  $(RH_p)$ .

**Lemma 6.2.** *Assume that the metric measure space  $(X, d, \mu, \mathcal{E})$  satisfies  $(D)$ .*

- (i) *If  $(P_2)$  holds, then there exists  $\varepsilon > 0$ , such that  $(RH_p)$  holds for each  $p \in (2, 2 + \varepsilon)$ .*
- (ii) *If there exists  $p_0 \in (2, \infty)$  such that  $(P_{p_0})$  and  $(RH_{p_0})$  holds, then there exists  $\varepsilon_1 > 0$  such that  $(RH_p)$  holds for each  $p \in (2, p_0 + \varepsilon_1)$ .*

*Proof.* (i) By [74], we know that  $(P_2)$  implies  $(P_{2-\tilde{\varepsilon}})$  for some  $0 < \tilde{\varepsilon} < 1$ . Therefore, for each  $u \in W^{1,2}(2B)$  satisfies  $\mathcal{L}u = 0$  in  $2B$ ,  $B = B(x_0, r)$ , we have

$$\left( \int_B |\nabla(u - u_{2B})|^2 d\mu \right)^{1/2} \leq \frac{C}{r} \int_{2B} |u - u_{2B}| d\mu \leq C \left( \int_{2B} |\nabla u|^{2-\tilde{\varepsilon}} d\mu \right)^{1/(2-\tilde{\varepsilon})}.$$

By applying the Gehring Lemma (cf. [51, 65]), we see that there exists  $\varepsilon > 0$  such that it holds for each  $p \in (2, 2 + \varepsilon)$

$$\left( \int_{B(x_0, r/2)} |\nabla u|^p d\mu \right)^{1/p} \leq C \left( \int_{B(x_0, r)} |\nabla u|^{2-\tilde{\varepsilon}} d\mu \right)^{1/(2-\tilde{\varepsilon})} \leq C \left( \int_{B(x_0, r)} |\nabla u|^2 d\mu \right)^{1/2}$$

$$\leq \frac{C}{r} \int_{2B} |u| d\mu.$$

Applying the geometric doubling lemma, Lemma 3.9, as in Step 4 of the proof of Theorem 3.5, we can conclude that  $(RH_p)$  holds for each  $p \in (2, 2 + \varepsilon)$ .

(ii) The second statement follows by noticing that,  $(P_{p_0})$  implies  $(P_{p_0-\hat{\varepsilon}})$  for some  $\hat{\varepsilon} > 0$ . This and  $(RH_{p_0})$  implies

$$\left( \int_B |\nabla u|^{p_0} d\mu \right)^{1/p_0} = \left( \int_B |\nabla(u - u_{2B})|^{p_0} d\mu \right)^{1/p_0} \leq \frac{C}{r} \int_{2B} |u - u_{2B}| d\mu \leq \left( \int_{2B} |\nabla u|^{p_0-\hat{\varepsilon}} d\mu \right)^{1/(p_0-\hat{\varepsilon})},$$

if  $u \in W^{1,2}(2B)$  satisfies  $\mathcal{L}u = 0$  in  $2B$ ,  $B = B(x_0, r)$ .

Using the Gehring Lemma (cf. [51, 65]) once more gives that, there exists  $\varepsilon_1 > 0$  such that  $(RH_p)$  holds for each  $p \in (2, p_0 + \varepsilon_1)$ .  $\square$

We can now prove Theorem 1.9 by using our main Theorem 1.6 and the above lemma.

*Proof of Theorem 1.9.* Notice that under the assumption of  $(D)$ ,  $(UE)$  and  $(P_p)$ ,  $(RH_p)$  or  $(G_p)$  implies  $(P_2)$ ; see [16, Corollary 2.8] and [15, Theorem 6.3]. The equivalence of  $(RH_p)$  and  $(G_p)$  follows from Corollary 1.7, we only need to prove that  $(G_p) \iff (R_p)$ .

**Step 1.**  $(R_p) \implies (G_p)$ .

We recall the argument for the sake of completeness. Assume  $(R_p)$ . By analyticity of the heat semigroup on  $L^p(X, \mu)$  (cf. [97])

$$\|\mathcal{L}^{1/2} e^{-t\mathcal{L}}\|_{p \rightarrow p} \leq \frac{C}{\sqrt{t}}.$$

Therefore, we can conclude that

$$\|\nabla H_t\|_{p \rightarrow p} = \|\nabla \mathcal{L}^{-1/2} \mathcal{L}^{1/2} H_t\|_{p \rightarrow p} \leq \frac{C}{\sqrt{t}},$$

i.e.,  $(G_p)$  holds.

**Step 2.**  $(G_p) \implies (R_p)$ .

Suppose that  $(G_p)$  holds. According to Corollary 1.7, we know that  $(RH_p)$  holds. By Lemma 6.2, there exists  $\varepsilon_1 > 0$  such that  $(RH_q)$  holds for each  $q \in (2, p + \varepsilon_1)$ . This, together with our main Theorem 1.6 and Theorem 6.1 above, yields that  $(R_q)$  holds for each  $q \in (2, p + \varepsilon_1)$ , and in particular,  $(R_p)$  holds, as desired.  $\square$

Corollary 1.10 now easily follows from Lemma 6.2 and Theorem 1.9.

*Proof of Corollary 1.10.* This corollary follows by combining Theorem 1.9 and Lemma 6.2.  $\square$

**Remark 6.3.** We remark that Theorem 1.9 and Corollary 1.10 admit localisations. Indeed, under local doubling and local Poincaré, the authors in [7] have shown that,  $(G_{p,\text{loc}})$  holds for all  $p \in (2, p_0)$  for some  $p_0 \in (2, \infty]$ , if and only if the local Riesz transform  $|\nabla(\mathcal{L} + a)^{-1/2}|$  is bounded on  $L^p(X, \mu)$  for all  $p \in (2, p_0)$ , where  $a > 0$  is a constant. Using our Theorem 5.3, it is then easy to see that  $(G_{p,\text{loc}})$ ,  $(RH_{p,\text{loc}})$  and the  $L^p$ -boundedness of the local Riesz transform are equivalent point-to-point for each  $p \in (2, \infty)$ . We leave the details to interested readers.

## 7 Sobolev inequality and isoperimetric inequality

In this section, following the idea of [70, 71] and using Theorem 3.13, we show that  $(RH_p)$  ( $p > 2$ ) yields a Sobolev inequality or isoperimetric inequality. Combining this and Theorem 1.9, we find a new necessary condition for quantitative regularity of harmonic functions, heat kernels and boundedness of Riesz transforms.

### 7.1 Sobolev inequality

For  $p \in (1, 2)$ , we say that the Sobolev inequality  $(S_{q,p})$  holds if there exists  $C = C(p, q)$  so that

$$\left( \int_B |f|^q d\mu \right)^{1/q} \leq Cr \left( \int_B |\nabla f|^p d\mu \right)^{1/p},$$

whenever  $B = B(x_0, r)$  is a ball and  $f$  is a Lipschitz function, compactly supported in  $B$ .

**Theorem 7.1.** *Let  $(X, d, \mu, \mathcal{E})$  be a metric measure space that satisfies  $(D_Q)$ ,  $Q > 2$ ,  $(UE)$  and  $(P_{2,\text{loc}})$ . Let  $p_0 \in (2, \infty)$ . Suppose that one of the following equivalent conditions,  $(RH_{p_0})$ ,  $(GLY_{p_0})$ ,  $(G_{p_0})$ ,  $(R_{p_0})$ , holds. Then for all  $q \in [\frac{p_0}{p_0-1}, 2]$  and  $s \in [1, \frac{qQ}{Q-q})$ , the Sobolev inequality  $(S_{s,q})$  holds.*

*Proof.* Let  $q_0 = \frac{p_0}{p_0-1}$  and take any  $t > \frac{Qp_0}{Q+p_0}$ . For each  $g \in L^\infty(B)$ , let  $f \in W_0^{1,2}(B(x_0, 2r))$  be the solution to  $\mathcal{L}f = g$  in  $B(x_0, 2r)$ . Then by Theorem 3.13, we can conclude that for each Lipschitz function  $v$ , compactly supported in  $B$ ,

$$\begin{aligned} \left| \int_B v(x)g(x) d\mu \right| &= \left| \int_B \langle \nabla v, \nabla f \rangle d\mu(x) \right| \\ &\leq C \|\nabla v\|_{L^{q_0}(B)} \|\nabla f\|_{L^{p_0}(B)} \\ &\leq C \|\nabla v\|_{L^{q_0}(B)} \frac{V(x_0, 2r)^{1/p_0}}{r} \left( \int_{B(x_0, 2r)} |f| d\mu + r^2 \left( \int_{B(x_0, 2r)} |g|^t d\mu \right)^{1/t} \right) \\ &\leq C \|\nabla v\|_{L^{q_0}(B)} r V(x_0, 2r)^{1/p_0} \left( \int_{B(x_0, 2r)} |g|^t d\mu \right)^{1/t}. \end{aligned}$$

Taking the supremum over  $g$  with  $\|g\|_{L^t(B)} \leq 1$  yields

$$\left( \int_B |v|^s d\mu \right)^{1/s} \leq Cr \left( \int_B |\nabla v|^{q_0} d\mu \right)^{1/q_0},$$

for any  $s < \frac{Qp_0}{Qp_0-Q-p_0} = \frac{Qq_0}{Q-q_0}$ . The Sobolev inequality  $(S_{s,q_0})$  then implies  $(S_{s,q})$  for any  $q \in [q_0, 2]$  and  $s \in [1, \frac{Qq}{Q-q})$ , as desired.  $\square$

## 7.2 Isoperimetric inequality

In this section, we give an application of Theorem 1.2 to isoperimetric inequalities. The following definition of perimeter can be found in [2, 85].

For an open set  $\Omega \subset X$ , denote by  $\text{Lip}(\Omega)$  ( $\text{Lip}_{\text{loc}}(\Omega)$ ) the space of all (locally) Lipschitz functions on  $\Omega$ , and  $\text{Lip}_0(\Omega)$  the space of all Lipschitz functions with compact support in  $\Omega$ . Denote by  $\mathcal{B}(X)$  the collection of all Borel sets in  $X$ .

**Definition 7.2.** Let  $E \in \mathcal{B}(X)$  and  $\Omega \subset X$  open. The perimeter of  $E$  in  $\Omega$ , denoted by  $P(E, \Omega)$ , is defined by

$$(7.1) \quad P(E, \Omega) = \inf \left\{ \liminf_{h \rightarrow \infty} \int_{\Omega} |\text{Lip } v_h| d\mu : \{v_h\}_h \subset \text{Lip}_{\text{loc}}(\Omega), v_h \rightarrow \chi_E \text{ in } L^1_{\text{loc}}(\Omega) \right\}.$$

$E$  is a set of finite perimeter in  $X$  if  $P(E, X) < \infty$ .

**Theorem 7.3.** Let  $(X, d, \mu, \mathcal{E})$  be a metric measure space that satisfies  $(D_Q)$ ,  $Q > 1$ ,  $(P_{\infty, \text{loc}})$  and  $(UE)$ . Suppose that one of the following equivalent conditions,  $(RH_{\infty})$ ,  $(GLY_{\infty})$ ,  $(G_{\infty})$ ,  $(GBE)$ , holds. Then, for every bounded Borel set  $E$  and each  $x_0 \in E$ ,

$$\mu(E)^{1-\frac{1}{Q}} \leq C \frac{r}{[V(x_0, r)]^{1/Q}} P(E, X).$$

where we choose  $r > \text{diam}(E)$  such that  $E \subset B(x_0, r)$ .

*Proof.* By [76, Theorem 2.2], it holds for all  $f, v \in \text{Lip}_{\text{loc}}(X)$  that

$$|\langle \nabla f, \nabla v \rangle| \leq \text{Lip } f \cdot \text{Lip } v.$$

Then the conclusion follows by applying Theorem 1.2 and [71, Theorem 1.2].  $\square$

**Corollary 7.4.** Let  $(X, d, \mu)$  be a non-compact Riemannian manifold, that satisfies  $(D_Q)$ ,  $Q > 1$ , and  $(UE)$ . Suppose that one of the following equivalent conditions,  $(RH_{\infty})$ ,  $(GLY_{\infty})$ ,  $(G_{\infty})$ ,  $(GBE)$ , holds. Then, for every bounded Borel set  $E$  and each  $x_0 \in E$ ,

$$\mu(E)^{1-\frac{1}{Q}} \leq C \frac{r}{[V(x_0, r)]^{1/Q}} P(E, X).$$

where we choose  $r > \text{diam}(E)$  such that  $E \subset B(x_0, r)$ .

**Remark 7.5.** We remark that Theorem 7.1 and Theorem 7.3 also admit localisation. Since the arguments are the same as for the global versions, we leave them to interested readers.

## 8 Examples

In this section, we wish to apply our results to several concrete examples of interest. Notice that since our assumptions are quite mild ( $(D)$ ,  $(UE)$  and  $(P_{2, \text{loc}})$ ), our results have broad applications. Below we will mainly concentrate on three different settings, and we refer the readers to [4, 5, 7, 13, 18, 44, 45, 76, 105] for more examples.

## 8.1 Riemannian metric measure spaces

Let us begin with some examples arising from Riemannian geometry.

**Example 1.** Riemannian metric measure spaces with Ricci curvature bounded from below, i.e.,  $RCD^*(K, N)$  spaces,  $K \in \mathbb{R}$  and  $N \in [1, \infty)$ ; see [5, 44]. Examples satisfying  $RCD^*(K, N)$  include complete Riemannian manifolds with dimension not bigger than  $N$  and Ricci curvature not less than  $K$ , and complete Alexandrov spaces with dimension not bigger than  $N$  and curvature not less than  $K$ . An important fact is that the  $RCD^*(K, N)$  condition is stable under Gromov-Hausdorff convergence, which means that a Gromov-Hausdorff limit, of a sequence of manifolds satisfying  $RCD^*(K, N)$ , satisfies also  $RCD^*(K, N)$ .

The  $RCD^*(K, N)$  condition can be defined as follows; see [5, 44]. Let  $(X, d, \mu, \mathcal{E})$  be a Dirichlet metric measure space satisfying  $\text{supp } \mu = X$  and  $V(x, r) \leq Ce^{cr^2}$  for some  $C, c > 0$ ,  $x \in X$  and each  $r > 0$ . Assume in addition that every  $f \in \mathcal{D}$  with  $|\nabla f| \leq 1$  has a 1-Lipschitz representative. We call  $(X, d, \mu, \mathcal{E})$  a  $RCD^*(K, N)$  space, where  $K \in \mathbb{R}$  and  $N \in [1, \infty)$ , if for all  $f \in \mathcal{D}$  and each  $t > 0$ , it holds

$$(8.1) \quad |\nabla H_t f(x)|^2 + \frac{4Kt^2}{N(e^{2Kt} - 1)} |\mathcal{L}H_t f(x)|^2 \leq e^{-2Kt} H_t(|\nabla f|^2)(x).$$

Equivalently,  $(X, d, \mu)$  is a  $RCD^*(K, N)$  space if the Cheeger energy is the associated Dirichlet form and  $CD^*(K, N)$  condition holds; see [5, 44].

Under  $RCD^*(K, N)$  condition, the (local) doubling condition was established in [84, 103], and the (local) Poincaré inequality was established in [90]. The doubling condition and Poincaré inequality have the same behavior as in the classical smooth manifolds.

Gradient estimates of harmonic functions and heat kernels on  $RCD^*(K, N)$  spaces were established in [50, 68, 69, 110]. Our results recover these gradient estimates in a more obvious and simple way. By the validity of the (local) doubling condition and (local) Poincaré inequality, the definition (8.1) implies directly  $(RH_\infty)$ ,  $(G_\infty)$  and  $(R_p)$  for all  $p \in (1, \infty)$  if  $K \geq 0$ , and their local versions if  $K < 0$ .

**Example 2.** On  $n$ -dimensional conical manifolds with compact basis without boundary,  $C(N) := \mathbb{R}^+ \times N$ , let  $\lambda_1$  be the smallest nonzero eigenvalue of the Laplacian on the basis  $N$  (see [30, 88] for studies on the first eigenvalue). By a result of Li [78], the Riesz transform is bounded on  $L^p(C(N))$  for all  $p \in (1, p_0)$  and not bounded for  $p \geq p_0$ , where

$$p_0 := n \left( \frac{n}{2} - \sqrt{\left( \frac{n-2}{2} \right)^2 + \lambda_1} \right)^{-1}$$

if  $\lambda_1 < n - 1$  and  $p_0 = \infty$  otherwise; see also [7].

Therefore by Theorem 1.9, we see that  $(RH_p)$  and  $(G_p)$  hold for all  $p < p_0$ . Moreover, if  $\lambda_1 < n - 1$ , then  $(RH_p)$  and  $(G_p)$  do not hold on  $C(N)$  for any  $p \geq p_0$ .

**Example 3.** By a result of Zhang [108], it is known that Yau's gradient estimate for harmonic functions is globally stable under certain perturbations of the metric in the following sense.

Let  $M$  be an  $n$ -dimensional Riemannian manifold,  $n > 2$ , suppose that the volume of each ball  $B(x, r)$  is comparable with  $r^n$  for any  $x \in M$  and  $r > 0$ , and assume that the  $L^2$ -Poincaré inequality holds. If

$$\text{Ric}(x) \geq -\frac{\varepsilon}{1 + d(x, x_0)^{2+\delta}}$$

for a fixed  $x_0 \in M$ ,  $\delta > 0$  and a sufficiently small  $\varepsilon > 0$ , then Yau's gradient estimate ( $Y_\infty$ ) holds with  $K = 0$ . This holds, in particular, if  $M$  is a small compact perturbation of a manifold of dimension at least 3 that has nonnegative Ricci curvature and maximum volume growth, i.e.,  $V(x, r) \sim r^n$ .

By Lemma 2.3, ( $Y_\infty$ ) with  $K = 0$  is equivalent to our ( $RH_\infty$ ). Therefore, by Theorem 1.2, we see that ( $RH_\infty$ ), ( $G_\infty$ ), ( $GLY_\infty$ ) and ( $GBE$ ) hold on these spaces.

**Example 4.** Let  $M$  be a Riemannian manifold that is the union of a compact part,  $M_0$ , and a finite number of Euclidean ends,  $\mathbb{R}^n \setminus B(0, 1)$ ,  $n \geq 3$ , each of which carries the standard metric. The volume of balls in  $M$  has the growth as  $V(x, r) \sim r^n$ , in particular, is a doubling measure. Moreover, ( $UE$ ) holds as a consequence of Sobolev inequality ( $LS_q$ ),  $q > 2$ . Notice also that, while ( $P_{2, \text{loc}}$ ) holds on  $M$ , ( $P_p$ ) does not hold for any  $p \leq n$ ; see [24, 33]. By [24], the Riesz transform is bounded on  $L^p(M)$  for  $p \in (2, n)$ , and is unbounded if  $p \geq n$ . Since ( $R_p$ ) implies ( $G_p$ ), Theorem 1.6 implies that ( $RH_p$ ) holds for all  $p < n$ .

Indeed, it is rather easy to see that ( $RH_p$ ) holds on  $M$  for  $p < n$ . Suppose that  $u$  is a harmonic function on  $2B$ ,  $B(x_0, r)$ . It is nothing to prove if  $r$  is small, since in this case, it holds

$$\|\nabla u\|_{L^\infty(B)} \leq \frac{C}{r} \int_B |u| d\mu.$$

If  $r \gg 1$ , then by using the pointwise Yau's gradient estimate ( $Y_\infty$ ) to  $u + \|u\|_{L^\infty(\frac{3}{2}B)}$ , we can conclude that

$$|\nabla u(x)| \leq \frac{C}{1 + \text{dist}(x, M_0)} \left( u(x) + \|u\|_{L^\infty(\frac{3}{2}B)} \right)$$

for each  $x \in B$ , which implies, if  $p < n$ ,

$$\left( \int_B |\nabla u|^p d\mu \right)^{1/p} \leq C \|u\|_{L^\infty(\frac{3}{2}B)} \left( \int_B \frac{1}{(1 + \text{dist}(x, M_0))^p} d\mu \right)^{1/p} \leq \frac{C}{r} \|u\|_{L^\infty(\frac{3}{2}B)} \leq \frac{C}{r} \int_{2B} |u| d\mu.$$

Notice that, however, ( $\widetilde{RH}_p$ ) does not hold on  $M$  for any  $p > 2$ . Indeed, if ( $\widetilde{RH}_p$ ) holds, then we have

$$\left( \int_B |\nabla u|^p d\mu \right)^{1/p} \leq C \mu(B)^{1/p-1/2} \left( \int_{2B} |\nabla u|^2 d\mu \right)^{1/2},$$

if  $u$  is harmonic on  $2B$ . By Li-Tam [81, Theorem 2.1], there exist a bounded, non-constant harmonic function  $u$  with finite Dirichlet energy. Applying the above estimate to  $u$  and letting the radius of  $B$  tend to infinity, we see that  $\|\nabla u\|_p = 0$ , which can not be true.



## 8.2 Carnot-Carathéodory spaces

A large class of examples that our results can be applied to come from Carnot-Carathéodory spaces; we refer the readers to [12, 13, 47, 54, 57, 66, 86] for background and recent developments.

Let  $M$  be a smooth, connected manifold and  $\mu$  a Borel measure. Let  $\{X_i\}_{i=1,\dots,m}$  be Lipschitz vector fields on  $M$ , with real coefficients. The “carré du champ” operator  $\Gamma$  is given as

$$\Gamma(f) := \sum_{i=1}^m |X_i f|^2$$

for each  $f \in C^\infty(M)$ , where the corresponding Dirichlet form  $\int_M \sum_i X_i f X_i g d\mu$  generalizes a second-order diffusion operator  $L$ .

A tangent vector  $v \in T_x M$  is called subunit for  $L$  at  $x$  if  $v = \sum_{i=1}^m a_i X_i(x)$ , with  $\sum_{i=1}^m a_i^2 \leq 1$ ; see [46]. A Lipschitz curve  $\gamma : [0, T] \mapsto M$  is called subunit for  $L$  if  $\gamma'(t)$  is subunit for  $L$  at  $\gamma(t)$  for a.e.  $t \in [0, T]$ . The subunit length of  $\gamma$ ,  $\ell(\gamma)$ , is given as  $T$ . We assume that for any  $x, y \in M$ , there always exists a subunit curve  $\gamma$  joining  $x$  to  $y$ . The Carnot-Carathéodory distance then is defined as

$$d_{cc}(p, q) := \inf\{\ell(\gamma) : \gamma \text{ is a subunit curve joining } p \text{ to } q\}.$$

Notice that for any  $x, y \in M$ , the Carnot-Carathéodory distance  $d_{cc}(x, y)$  is the same as  $d(x, y)$  induced from the Dirichlet forms; see [13, 23].

Once again, our results can be applied to this setting as soon as a (local) doubling condition and an (local)  $L^2$ -Poincaré inequality are available. Notice that all Carnot groups equipped with the Lebesgue measure and the natural vector fields satisfy an  $L^2$ -Poincaré inequality; see [57] for instance.

For general vector fields satisfying the Hörmander condition (cf. [46, 57, 66, 86]), it is known that the doubling condition and  $L^2$ -Poincaré inequality hold locally with constants depending on the balls under consideration, which is not sufficient in order to apply our results. However, the potential estimates for the Poisson equation from Section 3, Theorem 3.12 and Theorem 3.13, still work in these settings.

As we recalled in the introduction, by Theorem 1.2,  $(RH_\infty)$ ,  $(GLY_\infty)$ ,  $(G_\infty)$  and  $(GBE)$  hold on Heisenberg groups  $\mathbb{H}(2n, m)$  (cf. [37, 63]), and more generally, on sub-Riemannian manifolds satisfying Baudoin-Garofalo’s curvature-dimension inequality  $CD(\rho_1, \rho_2, \kappa, d)$  (cf. [13]) with  $\rho_1 \geq 0$ ,  $\rho_2 > 0$ ,  $\kappa \geq 0$  and  $d \geq 2$ .

Examples satisfying  $CD(\rho_1, \rho_2, \kappa, d)$  include all Sasakian manifolds whose horizontal Webster-Tanaka-Ricci curvature is bounded from below, all Carnot groups with step two, and wide subclasses of principal bundles over Riemannian manifolds whose Ricci curvature is bounded from below; see [13, Section 2].

## 8.3 Degenerate (sub-)elliptic/parabolic equations

Our results are also applicable to degenerate (sub-)elliptic/parabolic equations on Euclidean spaces. It is of course also possible to extend these degenerate equations to general metric measure

spaces. For instance, one may consider a Dirichlet form given by

$$\int_X \langle \nabla f(x) \cdot \nabla g(x) \rangle w(x) dx,$$

where  $\langle \nabla f, \nabla g \rangle$  is the natural density of energy on an infinitesimally Hilbertian space  $(X, d, \mu)$  (cf. [4]), or  $\nabla$  is the Cheeger differential operator (cf. [25]), and  $w$  is a suitable weight.

We focus on degenerate elliptic/parabolic equations on Euclidean spaces, and we refer the reader to [18, p.133] and [48] for more examples of degenerate (sub-)elliptic equations.

Let  $w$  be a Muckenhoupt  $A_2$ -weight, or  $w := |J_f|^{1-\frac{2}{n}}$ , where  $|J_f|$  denotes the Jacobian of a quasiconformal mapping  $f$  on  $\mathbb{R}^n$ ; see [18, 45]. Let  $A := (A_{ij}(x))_{i,j=1}^n$  be a symmetric matrix of functions on  $\mathbb{R}^n$  satisfying the *degenerate ellipticity condition*, namely, there exist constants  $0 < \lambda \leq \Lambda < \infty$  such that, for all  $\xi \in \mathbb{R}^n$ ,

$$(8.2) \quad \lambda w(x) |\xi|^2 \leq \langle A\xi, \xi \rangle \leq \Lambda w(x) |\xi|^2.$$

For all  $f, g \in C_c^\infty(\mathbb{R}^n)$ , consider the Dirichlet form given by

$$(8.3) \quad \int_{\mathbb{R}^n} A(x) \nabla f(x) \cdot \nabla g(x) dx.$$

Then the intrinsic distance  $d$  is equivalent to the usual Euclidean metric  $d_E$ . On the space  $(\mathbb{R}^n, d_E, w(x) dx)$ , the doubling condition is a well-known property of a Muckenhoupt weight or follows from properties of quasiconformal mappings, and an  $L^2$ -Poincaré inequality was established in [45]. From this, one can deduce that a doubling condition and a weak  $L^2$ -Poincaré inequality, i.e.

$$(8.4) \quad \oint_B |f - f_B| d\mu \leq Cr \left( \oint_{cB} |\nabla f|^2 d\mu \right)^{1/2},$$

for all  $B = B(x, r)$ , for some constant  $c \geq 1$ , hold on  $(\mathbb{R}^n, d, w(x) dx)$ . By using the results from [57, Section 9], together with the fact that  $(\mathbb{R}^n, d, w(x) dx)$  is geodesic, we see that  $(\mathbb{R}^n, d, w(x) dx)$  supports a scale-invariant  $L^2$ -Poincaré inequality.

Therefore, our results are applicable to  $(\mathbb{R}^n, d, w(x) dx)$  as well. We would like to point out that Caffarelli and Peral [22] established a  $W^{1,p}$  estimate for elliptic equations in divergence form by using the technique of approximation to a reference equation. Shen [95] employed the techniques from [22] to prove the equivalence of  $(R_p)$  and  $(RH_p)$ , for *uniformly elliptic* operators of divergence form on  $\mathbb{R}^n$ .

For degenerate equations, although the heat kernel and harmonic functions are known to be Hölder continuous (cf. [18, 99, 100, 101]), harmonic functions and the heat kernel are not Lipschitz in general; see the examples from the introductions of [68, 75] for instance.

Moreover, given an explicit  $p > 2$ , we do not even know if the gradients of harmonic functions or heat kernels are locally  $L^p$ -integrable. Indeed, in view of Corollary 1.10 and Theorem 1.9, we see that there exists  $\varepsilon > 0$  (implicit), such that  $(RH_p)$  and  $(G_p)$  hold for  $p \in (2, 2 + \varepsilon)$ . However,

for an explicitly given  $p > 2$ , the assumption  $w \in A_2$  alone is not sufficient for quantitative  $L^p$ -regularity of harmonic functions or heat kernels, in view of Theorem 7.1. Since if  $(RH_p)$  or  $(G_p)$  holds for some  $p > 2$ , then one has a Sobolev inequality  $(S_{p',q})$  for some  $q > p'$  on  $(\mathbb{R}^n, d, w(x) dx)$ , and it is well-known that  $w \in A_2$  is not sufficient to guarantee such a Sobolev inequality for (small)  $p'$ .

It would be of great interest to know how to quantify the regularity of harmonic functions and heat kernels in this case.

## 8.4 Covering manifolds

Consider a complete, non-compact, connected Riemannian manifold  $M$ . Suppose that a finitely generated discrete group  $G$  acts properly and freely on  $M$  by isometries, such that the orbit space  $M_1 = M/G$  is a compact manifold. In other words,  $M$  is a Galois covering manifold of the compact Riemannian manifold  $M_1$ , with deck transformation group (isomorphic to)  $G$ . The most simple example is  $M = \mathbb{R}^D$  endowed with a Riemannian metric which is periodic under the standard action of  $G = \mathbb{Z}^D$  by translations.

Assuming that  $G$  has polynomial volume growth of some order  $D \geq 1$ , Dungey [39, Theorem 1.1] (see also [38]) showed that  $(GLY_\infty)$  holds on  $M$ . Our Theorem 1.2 then implies that  $(RH_\infty)$ ,  $(G_\infty)$ ,  $(GLY_\infty)$  and  $(GBE)$  hold on these spaces.

In the following, we however wish to provide a direct proof of  $(RH_\infty)$  on  $M$ , which together with our Theorem 1.2 easily implies  $(G_\infty)$ . Notice that our proof below works well also for Lie groups of polynomial growth; see [1, 91].

**Theorem 8.1.** *Let  $M$  be as above. Then  $(RH_\infty)$  holds on  $M$ .*

Let us observe that, due to the group action and  $G$  has polynomial volume growth,  $(D)$  and  $(P_2)$  hold on  $M$ . Moreover,  $(RH_{\infty, \text{loc}})$  holds, i.e., for each  $r_0 > 1$ , there exists  $C(r_0) > 0$  such that if  $u$  is harmonic in  $2B$ ,  $B = B(x_0, r)$ ,  $r < r_0$ , it holds

$$(RH_{\infty, \text{loc}}) \quad \|\nabla u\|_{L^\infty(B)} \leq \frac{C(r_0)}{r} \int_{2B} |u| d\mu.$$

*Proof.* Since  $(RH_{\infty, \text{loc}})$  holds, we only need to prove  $(RH_\infty)$  for balls of large radii.

Since  $(D)$  and  $(P_2)$  hold, there exist  $C > 0$  and  $\gamma \in (0, 1)$ , such that for each ball  $B = B(x_0, r)$  and if  $u$  is harmonic on  $2B$ , it holds for all  $x, y \in B(x_0, r)$  that

$$(8.5) \quad |u(x) - u(y)| \leq C \frac{d^\gamma(x, y)}{r^\gamma} \int_{2B} |u| d\mu.$$

We may assume  $r \gg 1$ , and fix a fundamental domain  $X \subset B = B(x_0, r)$ . Then  $g \cdot X$  are pairwise disjoint for different  $g \in G$ , and  $M \setminus (G \cdot X)$  is of measure zero. For simplicity of notions we assume that  $u$  is harmonic on  $8B$ .

**Claim 1:** For each  $x \in X$ ,  $g \in G$  such that  $g \cdot x \in B(x_0, r)$ , it holds

$$(8.6) \quad |u(x) - u(g \cdot x)| \leq C \frac{\rho(g)}{r} \int_{4B} |u| d\mu.$$

*Proof of Claim 1.* If  $d(x, g \cdot x) \geq 2^{-16}r$ , then (8.6) is obvious. Consider now  $d(x, g \cdot x) < 2^{-16}r$ . Let  $k \in \mathbb{N}$  such that

$$2^k < r\rho(g)^{-1} \leq 2^{k+1}$$

(Remember  $r \gg 1$  and  $\rho(g) \geq 1$ ). For each  $j \in \{1, \dots, k\}$ , noticing that  $g^{2^j} \cdot x \in 2B$ , since  $d(g^{2^j} \cdot x, x) < r$ . By using (8.5) twice, we see that for  $1 \leq j \leq k$  it holds

$$\begin{aligned} \left| [u(x) - u(g^{2^j} \cdot x)] - [u(g^{2^j} \cdot x) - u(g^{2^{j+1}} \cdot x)] \right| &\leq C \frac{d^\gamma(x, g^{2^j} \cdot x)}{r^\gamma} \int_{3B} |u(x) - u(g^{2^j} \cdot x)| d\mu(x) \\ &\leq C \frac{2^{2j\gamma} \rho(g)^{2\gamma}}{r^{2\gamma}} \int_{4B} |u| d\mu. \end{aligned}$$

Notice it holds the identity

$$I - A = \sum_{j=0}^k 2^{-j-1} [1 - A^{2^j}]^2 + 2^{-k-1} (1 - A^{2^{k+1}}),$$

Using the identity together with the above estimate and  $2^k < r\rho(g)^{-1} \leq 2^{k+1}$ , we see that for  $\gamma < \beta < 2\gamma$  and  $\beta \leq 1$ , it holds

$$\begin{aligned} |u(x) - u(g \cdot x)| &\leq \sum_{j=0}^k 2^{-j-1} |u(x) - 2u(g^{2^j} \cdot x) + u(g^{2^{j+1}} \cdot x)| + 2^{-k-1} |u(x) - u(g^{2^{k+1}} \cdot x)| \\ &\leq \sum_{j=0}^k C 2^{-j-1} \frac{2^{2j\gamma} \rho(g)^{2\gamma}}{r^{2\gamma}} \int_{4B} |u| d\mu + C 2^{-k-1} \frac{2^{k\gamma} \rho(g)^\gamma}{r^\gamma} \int_{4B} |u| d\mu \\ &\leq C \int_{4B} |u| d\mu \left( \sum_{j=0}^k 2^{-j-1+2j\gamma-2k\gamma} + 2^{-k-1} \right) \\ &\leq C \int_{4B} |u| d\mu \left( \sum_{j=0}^k 2^{j(2\gamma-1)-k(2\gamma-\beta)} 2^{-k\beta} + 2^{-k-1} \right) \\ &\leq C 2^{-k\beta} \int_{4B} |u| d\mu \leq C \frac{\rho(g)^\beta}{r^\beta} \int_{4B} |u| d\mu. \end{aligned}$$

Repeating this argument finite times, we can conclude that (8.6) holds.

**Claim 2:** There exists a finite set  $J \subset G$  of finite elements, with  $e \in J$ , such that

$$\sup_{x, y \in \sup_{g \in J} g \cdot X} |u(x) - u(y)| \leq C \sup_{g \in J, x \in X} |u(x) - u(g \cdot x)|$$

*Proof of Claim 2.* Take  $y_0 \in X$  and fix  $0 < r_1$  such that  $X \subset B(y_0, r_1)$ . Then by Proposition 2.1, there exists  $r_2 > r_1$  such that

$$(8.7) \quad \sup_{x, y \in B(y_0, r_1)} |u(x) - u(y)| \leq \frac{1}{2} \sup_{x, y \in B(y_0, r_2)} |u(x) - u(y)|.$$

Let  $J \subset G$  be the collection of  $g \in G$  such that  $g \cdot X \cap B(y_0, r_2) \neq \emptyset$ . Then  $B(y_0, r_2) \subset \cup_{g \in J} g \cdot X$  and  $J$  only has finite elements. For  $x, y \in \cup_{g \in J} g \cdot X$ , take  $\tilde{x}, \tilde{y} \in X$  and  $g, h \in G$  such that  $x = g \cdot \tilde{x}$  and  $y = h \cdot \tilde{y}$ . Then by (8.7) we obtain

$$\begin{aligned} |u(x) - u(y)| &\leq |u(g \cdot \tilde{x}) - u(\tilde{x})| + |u(h \cdot \tilde{y}) - u(\tilde{y})| + |u(\tilde{x}) - u(\tilde{y})| \\ &\leq 2 \sup_{g \in J, x \in X} |u(x) - u(g \cdot x)| + \sup_{\tilde{x}, \tilde{y} \in X} |u(\tilde{x}) - u(\tilde{y})| \\ &\leq 2 \sup_{g \in J, x \in X} |u(x) - u(g \cdot x)| + \frac{1}{2} \sup_{\tilde{x}, \tilde{y} \in B(y_0, r_2)} |u(\tilde{x}) - u(\tilde{y})|, \end{aligned}$$

which, together with the fact  $B(y_0, r_2) \subset \cup_{g \in J} g \cdot X$ , implies that

$$\sup_{x, y \in \sup_{g \in J} g \cdot X} |u(x) - u(y)| \leq C \sup_{g \in J, x \in X} |u(x) - u(g \cdot x)|.$$

We can now complete the proof.

Recall that  $y_0 \in X \subset B$ ,  $X \subset B(x_0, r_1) \subset B(x_0, r_2) \subset \cup_{g \in J} g \cdot X$ . Fix  $r_3 > r_2$  such that  $\cup_{g \in J} g \cdot X \subset B(x_0, r_3)$ . Notice  $J \subset G$  is a fixed finite set.

Since  $(RH_{\infty, \text{loc}})$  holds, we may assume that  $r > r_1 + r_3$  large enough. Then for each  $h \in \{h \in G : h \cdot X \cap B \neq \emptyset\}$ ,  $hg \cdot X \subset 2B$  for each  $g \in J$ . By  $(RH_{\infty, \text{loc}})$ , together with the previous two claims, we obtain for each  $h \in \{h \in G : h \cdot X \cap B \neq \emptyset\}$ ,

$$\begin{aligned} |||\nabla u|||_{L^\infty(h \cdot X)} &\leq |||\nabla(u - u(h \cdot y_0))|||_{L^\infty(B(h \cdot y_0, r_1))} \leq C \int_{B(h \cdot y_0, r_2)} |u - u(h \cdot y_0)| d\mu \\ &\leq C \int_{\cup_{g \in J} hg \cdot X} |u - u(h \cdot y_0)| d\mu \leq C \sup_{x, y \in \cup_{g \in J} gh \cdot X} |u(x) - u(y)| \\ &\leq C \sup_{g \in J, x \in X} |u(h \cdot x) - u(hg \cdot x)| \leq \frac{C}{r} \int_{8B} |u| d\mu. \end{aligned}$$

This implies that

$$|||\nabla u|||_{L^\infty(B)} \leq \sup_{h \in G : h \cdot X \cap B \neq \emptyset} |||\nabla u|||_{L^\infty(h \cdot X)} \leq \frac{C}{r} \int_{8B} |u| d\mu.$$

A covering argument similar to that of **Step 4** in the proof of Theorem 3.5 then gives  $(RH_\infty)$ .  $\square$

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Renjin Jiang

Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China

Thierry Coulhon

PSL Research University, 75005 Paris, France.

Pekka Koskela

Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014, Finland.

Adam Sikora

Department of Mathematics, Macquarie University, NSW 2109, Australia.

*E-mail addresses:* [rejiang@tju.edu.cn](mailto:rejiang@tju.edu.cn)

[thierry.coulhon@univ-psl.fr](mailto:thierry.coulhon@univ-psl.fr)

[pekka.j.koskela@jyu.fi](mailto:pekka.j.koskela@jyu.fi)

[adam.sikora@mq.edu.au](mailto:adam.sikora@mq.edu.au)